மனோன்மணியம் சுந்தரனார் பல்கலைக்கழகம்

MANONMANIAM SUNDARANAR UNIVERSITY

TIRUNELVELI – 12

தொலைநிலை தொடர்கல்வி இயக்ககம்

DIRECTORATE OF DISTANCE & CONTINUING EDUCATION

BBA - FIRST YEAR

BUSINESS MATHEMATICS

B.B.A. - FIRST YEAR BUSINESS MATHEMATICS SYLLABUS

Course Objective:

To acquaint students with the construction of mathematical models for Managerial decision situations. The emphasis is on understanding the concepts, formulation and interpretation.

UNIT- I: ANALYTICAL GEOMETRY

Analytical geometry – distance between two points in a plane – slope of a straight line - equation of the straight line – point of intersection – demand and supply curves (linear) – market equilibrium – break even analysis.

UNIT –II: SET THEORY

Set theory – definition – types – union, intersection, difference, and complement of sets De Morgan's Law – Venn diagram – simple set applications – Cartesian product.

UNIT- III: DIFFERENTIAL CALCULUS

Differential calculus – derivative of a function – differentiation – standard forms – sum, product, quotient rule – differential coefficients of simple functions (trigonometric functions excluded) – function of a function rule – simple application to business using marginal concept.

UNIT-IV: HIGHER ORDER DERIVATIVES

Higher order derivatives – maxima and minima – simple marketing models using profit maximization, fencing and container problems only – Integral calculus – standard forms – rules of integration – Definite integral – simple applications – finding total and average cost function – producer surplus and consumer surplus.

UNIT -V: MATRIX

Matrices – definition – types – addition, subtraction, multiplication of matrices – inverse matrix – solving a system of simultaneous linear equations using matrix inversion technique – rank of a matrix.

(Marks: Theory 40% and Problems 60%)

Reference Books:

1. Business Mathematics – V.Sundaresan and S.D.Jeyaseelan.

2. Business Mathematics – Navaneethan .P

3. Business Mathematics - M. Wilson

4. Mathematics for Management – G.k. Ranganath

UNIT - I

ANALYTICAL GEOMETRY

Analytical geometry – distance between two points in a plane – slope of a straight line - equation of the straight line – point of intersection – demand and supply curves (linear) – market equilibrium – break even analysis.

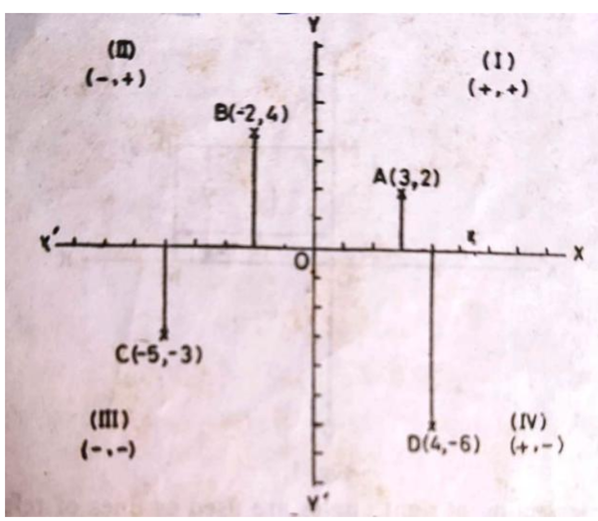
Introduction:

Analytical geometry is a branch of mathematics in which algebraic methods are employed to solve the problems in Geometry. It was the French mathematician Rane de Cartes who first introduced this method. It means geometry of analytic varieties. It is also referred to as coordinate geometry or Cartesian geometry.

Coordinate Axes:

Any two straight lines in the plane intersecting at right angles are used as lines of references and a point is located in the plane by giving its distance from each of them. The lines of reference are called the **Coordinate Axes** or briefly the axes and the point of intersection, the **Origin**. We take the lines to be horizontal (X-axis) and vertical (Y-axis).

The plane is divided into four parts known as quadrants. The x-Coordinate (also called abscissa) of a point is its distance from its y-axis and it can be taken positive if it is on the right and negative if it is on the left of the y-axis. The y-coordinate (also called ordinate) of the point is its distance from the x-axis and it is considered positive if it is above and negative if below the x-axis.

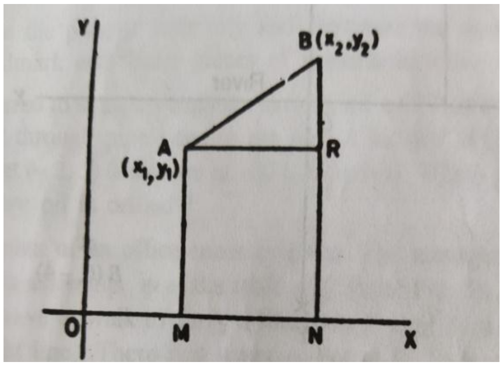


The point (2,3) is different from the point (3, 2). The points having both the coordinates as positive numbers lie in the I quadrant. When both the co-ordinates of a point are negative the point is in the III quadrant. If the x-coordinate of a point is negative and the y-coordinate of the point is positive then the point is in the II quadrant. For a point in the IV quadrant the x-coordinate is positive and y-coordinate is negative.

Distance between Two points in a plane :

We establish a formula for finding the distance between two points.

Let A(x_1, y_1) and B (x_2, y_2) be two points. For convenience we take the points in the first quadrant and establish the formula. The formula will be true whether A and B may be. We draw perpendiculars AM, BN to the *x*-axis and AR perpendicular to BN.



$$AR = MN = ON - OM = x_2 - x_1$$

$$BR = BN - RN = BN - AM = y_2 - y_1$$

Δ ABR is a right angle triangle and by the Pythagoreous Theorem,

$$AB^2 = AR^2 + BR^2$$

$$AB = \sqrt{AR^2 + BR^2}$$

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 1

If A (-3,3), B (5,9), and C (-7,4) find the distance between A and B ; B and C.

Solution:

Distance between two points AB = $\sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$

A = (-3,3)   B = (5,9);

$x_2 = 5$, $x_1 = -3$, $y_2 = 9$, $y_1 = 3$

AB = $\sqrt{\{5-(-3)\}^2 + (9-3)^2}$

= $\sqrt{(8)^2 + (6)^2}$

= $\sqrt{64+36}$

= $\sqrt{100}$

= 10

B = (5,9); C = (-7,4);

$x_2 = -7$, $x_1 = 5$, $y_2 = 4$, $y_1 = 9$

BC = $\sqrt{(-7-5)^2 + (4-9)^2}$

= $\sqrt{(-12)^2 + (-5)^2}$

= $\sqrt{144+25}$

= $\sqrt{169}$

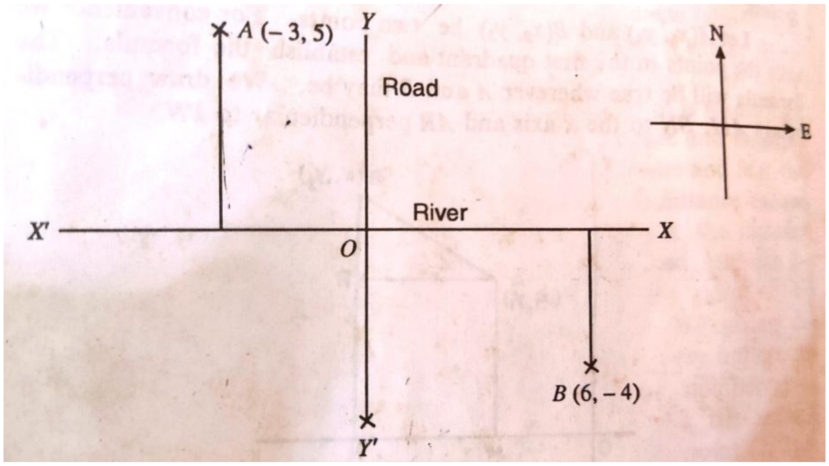
= 13.

Example 2

A road runs north-south and a river runs east-west. A factory is 5 kms. north of the river and 3 kms. west of the road. Another factory is 4 kms. south of the river and 6 kms. east of the road. Find the length of the telephone line connecting the two factories.

Solution:

Let us take the river as the x-axis and the road as the y-axis. The point O where they cross is the origin. Let the first factory be A and the second be B.



The factories are represented by the points A (-3,5) and B (6,-4).

Distance between two points AB = $\sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$

$$AB = \sqrt{\{6-(-3)\}^2 + (-4-5)^2}$$

$$= \sqrt{(9)^2 + (-9)^2}$$

$$= \sqrt{81+81}$$

$$= \sqrt{162}$$

= 12.73 Kms.

Example 3:

Prove that the points P(1,1), Q(-1,-1) and R $\left(-\sqrt{3}, \sqrt{3}\right)$ are the vertices of an equilateral triangle.

Solution:

Distance between two points $AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$PQ^2 = (1+1)^2 + (1+1)^2$

$= 4+4 = 8$

$QR^2 = \left(-1+\sqrt{3}\right)^2 + \left(-1-\sqrt{3}\right)^2$ $(a + b)^2 = a^2 + b^2 + 2ab$

$= [1 + 3 - 2\sqrt{3}\,] + [1 + 3 + 2\sqrt{3}] = 8$ $(a - b)^2 = a^2 + b^2 - 2ab$

$RP^2 = \left(-\sqrt{3} - 1\right)^2 + \left(\sqrt{3} - 1\right)^2$

$= [3 + 1 + 2\sqrt{3}\,] + [3 + 1 - 2\sqrt{3}] = 8$

PQ = QR = RP

i.e., all the sides of the triangle are equal.

Therefore, the triangle is equilateral triangle.

Section formula:

The point dividing the line joining A (x_1, y_1) and B (x_2, y_2) in the ratio $l : m$ is given by

$$P\left(\frac{lx_2 + mx_1}{l+m}, \frac{ly_2 + my_1}{l+m}\right)$$

If P is the midpoint of AB the $l = m$

Therefore P is given by $P\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

If P divides AB externally P is given by $P\left(\frac{lx_2 - mx_1}{l-m}, \frac{ly_2 - my_1}{l-m}\right)$

Example

Find the point which divides the line joining A(2,3) and B(12,18) in the ratio 2 : 3. What is the midpoint of AB?

Solution :

Let $P(x,y)$ divide AB in the ratio 2 : 3 using section formula.

$$P\left(\frac{lx_2+mx_1}{l+m},\frac{ly_2+my_1}{l+m}\right)$$

Where,

$x_1 = 2$ $\quad y_1 = 3$ $\quad x_2 = 12$ $\quad y_2 = 18$ $\quad l = 2$ $\quad m = 3$

$$P\left(\frac{2\times12+3\times2}{2+3},\frac{2\times18+3\times3}{2+3}\right)$$

i.e., P (6,9)

Let M be the midpoint of AB

Then $M\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

i.e., $M\left(\frac{2+12}{2},\frac{3+18}{2}\right)$

i.e., $M\left(7,\frac{21}{2}\right)$

Slope of a Straight Line or Gradient of a Straight Line:

Slope of the line AB = $\frac{y_2-y_1}{x_2-x_1}$

Slope of the line AB = $\frac{y\ co-ordinate\ \ of\ A-y\ co-ordinate\ \ of\ B}{x\ co-ordinate\ \ of\ A-x\ co-ordinate\ \ of\ B}$

Example

Find the slope of the line joining P(-2,3) and Q(8,-5)

Solution:

Slope of the line PQ = $\frac{y_2-y_1}{x_2-x_1}$

$= \frac{-5-3}{8-(-2)}$

$= \frac{-8}{10}$

$= -\frac{4}{5}$

Equation of a Straight Line and Applications:

i. Slope – intercept form:

Let a straight line, whose slope is m, cut off an intercept OB = c along the y-axis, Let *P(x,y)* be any point on the line.

Slope = $\frac{Rise}{Run}$

$\boldsymbol{y = mx+c}$

This relation $y = mx+c$ between the co-ordinates (x,y) of any point P is called the equation of the straight line. The co-ordinates of every point on the line will satisfy the equation $y = mx+c$, Whereas the co-ordinates of any point not on the line will not satisfy the equation $y = mx+c$.

For example,

$y = 3x+8$ is the equation of the line whose slope is 3 and *y*- intercept is 8. (intercept made on the Y-axis) The point (1,11) is a point on the line, for, when we put x = 1 and y = 11, the above equation is satisfied. But (2,4) is not a point on the line. For, L.H.S. = 4 and R.H.S = $3\times 2 + 8$ =14 and $4 \neq 14$.

Note: If the *y* – intercept is below the *x*-axis it will be a negative quantity

i.e., $c < 0$, if the *y*-intercept *c* lies below the X-axis.

Example: 1

Find the equation of the line whose slope is $\frac{3}{2}$ and which cuts off 3 units along OY.

Solution :

Here, m = $\frac{3}{2}$ and c = + 3

Equation of the line is, $y = mx+c$

$$y = \frac{3}{2}x + 3$$

$$2y = 3x + 6$$

$$3x - 2y + 6 = 0$$

ii. Point – slope form:

If the line passes through a given fixed point $A(x_1, y_1)$ and has a given slope m, to find its equation.

Let $P(x,y)$ be any point on the line.

$\boldsymbol{y - y_1 = m\,(x - x_1)}$ gives the equation of the straight line.

Example: 2

Find the equation of the line passing through the point (2,-3) having the slope $\frac{-5}{7}$

Solution :

The equation of the line is, $\boldsymbol{y - y_1 = m\,(x - x_1)}$

$$y - (-3) = \frac{-5}{7}(x - 2)$$

$$y + 3 = \frac{-5}{7}(x - 2)$$

$$7(y + 3) = -5(x - 2)$$

$$7y + 21 = -5x + 10$$

$$5x + 7y + 11 = 0.$$

iii. Two point form:

It is given that a line passes through two points, $A(x_1, y_1)$ and $B(x_2, y_2)$.

Let $P(x,y)$ be any point on the line.

The slope of the line = $\frac{y_1 - y_2}{x_1 - x_2}$

$\boldsymbol{y - y_1} = \frac{\boldsymbol{y_1 - y_2}}{\boldsymbol{x_1 - x_2}}\left(\boldsymbol{x - x_1}\right)$ gives the equation of the straight line joining two given points $A\ (x_1, y_1)$ and $B\ (x_2, y_2)$.

Example: 3

Find the equation of the line passing through (2,-3) and (-4,5)

Solution :

The equation of the line is, $\boldsymbol{y - y_1} = \frac{\boldsymbol{y_1 - y_2}}{\boldsymbol{x_1 - x_2}}\left(\boldsymbol{x - x_1}\right)$

$$y - (-3) = \frac{-3-5}{2-(-4)}(x - 2)$$

$$y + 3 = \frac{-8}{6}(x - 2)$$

$$6y + 18 = -8x + 16$$

$$8x + 6y + 2 = 0$$

$$4x + 3y + 1 = 0$$

All the four equations we got above are first degree equations it x and y. In fact any first degree equation in x and y represents a straight line in a plane. That is why any relationship between two variables x and y expressed as a first degree equation in x and y is called linear relationship.

iv. Intercept form:

The two-point form, the equation of the line AB is,

$$\frac{x}{a} + \frac{y}{b} = 1$$

Example: 4

A line passes through (-3, 10) and sum of its intercepts on axes is 8. Find the equation.

Solution :

Let the intercept made on the X-axis be a. Then the intercept on the Y-axis is $8 - a$.

$a + b = 8$

$b = 8 - a$

Equation of the line is, $\frac{x}{a} + \frac{y}{b} = 1$

$\frac{x}{a} + \frac{y}{8-a} = 1$

The line passes through (-3, 10)

$\frac{-3}{a} + \frac{10}{8-a} = 1$

$\frac{-3(8-a)+10a}{a(8-a)} = 1$

$-24 + 3a + 10a = 8a - a^2$

$a^2 + 5a - 24 = 0$

$(a + 8)(a - 3) = 0$

$a = -8$ or 3

Taking a = 3, the equation of the line is

$\frac{x}{3} + \frac{y}{5} = 1$

$5x + 3y = 15$

Taking a = -8, the equation of the line is

$\frac{x}{-8} + \frac{y}{16} = 1$

$-2x + y = 16$

$2x - y + 16 = 0$

Thus we have two lines.

Example: 5

As the number of units manufactured increases from 2000 to 3000 the total cost of production increases from Rs. 11,000 to Rs. 15,000. Find the relationship between the cost (y) and the number of units made (x), if the relationship is linear.

Solution :

As the relationship between x and y is linear we have to find the line through (2000, 11,000) and (3000, 15,000)

When $x_1 = 2000$ $y_1 = 11{,}000$ and

$x_2 = 3000$ $y_2 = 15{,}000$

The required relationship is,

$$\boldsymbol{y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}\left(x - x_1\right)}$$

$$y - 11000 = \frac{11000 - 15000}{2000 - 3000}(x - 2000)$$

$$y - 11000 = 4\,(x - 2000)$$

$$4x - y + 3000 = 0.$$

Example: 6

Mr. Ram buys a radio making a payment of Rs. 200 at the time of purchase with the agreement that he will pay at the rate of Rs. 15 for the next 20 months. Find the relationship between the amount (y) he has paid and the number of months (x) since he bought the radio.

Solution :

From the problem it is evident that when $x = 0$, $y = 200$.

Thus the y - intercept is 200

i.e., $c = 200$.

For a unit change in x the change in y is 15

i.e., the vertical change is 15 for a unit horizontal change.

Therefore the slope of the line is 15.

Hence the required relationship is,

$$y = \mathrm{m}x + c$$

$$y = 15x + 200.$$

Example: 7

A straight line passes through the point (-4,9) and is such that the portion of it intercepted between the axes divided at the point in the ratio 3: 2. Find the equation.

Solution:

Let the line meet the X-axis at A(a, 0) and Y-axis at B(0, b)

i.e., AB is the portion of the line intercepted between the axes.

Given that the point P(-4, 9) divides AB where A(a , 0) and B(0, b) in the ratio 3 : 2

$$P\left(\frac{lx_2 + mx_1}{l+m}, \frac{ly_2 + my_1}{l+m}\right)$$

$$-4 = \frac{3.0+2.a}{3+2} = \frac{2a}{5} \qquad ...(1)$$

$$9 = \frac{3.b+2.0}{3+2} = \frac{3b}{5} \qquad ...(2)$$

From (1) $a = -10$

From (2) $b = 15$

Equation of the line AB is,

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\frac{x}{-10} + \frac{y}{15} = 1$$

$$-3x + 2y = 30$$

$$3x - 2y + 30 = 0.$$

Example 8

Find the ratio in which the join (-5,1) and (1,-3) divides the straight line passing through (3,4) and (7,8).

Solution :

Let the points be denoted by A(-5,1), B(1,-3), P(3,4) and Q(7,8).

Let the line joining AB meet the line joining PQ at R.

We have to find the ratio $\frac{PR}{RQ}$.

Let $\frac{PR}{RQ} = \frac{l}{m}$, and the co-ordinates of R be (x, y)

Then $x = \frac{7l+3m}{l+m}$,

$y = \frac{8l+4m}{l+m}$ using the section formula $P\left(\frac{lx_2+mx_1}{l+m}, \frac{ly_2+my_1}{l+m}\right)$

Equation of the line AB is

$$\boldsymbol{y - y_1 = \frac{y_1-y_2}{x_1-x_2}\left(x - x_1\right)}$$

$$y - 1 = \frac{1+3}{-5-1}(x+5)$$

$$y - 1 = \frac{-2}{3}(x+5)$$

$$3y - 3 = -2x - 10$$

$$2x + 3y + 7 = 0$$

R lies on AB.

The co-ordinates of R satisfy the equation of AB.

$$2\left(\frac{7l+3m}{l+m}\right)+3\left(\frac{8l+4m}{l+m}\right)+7 = 0$$

$$14l + 6m + 24l + 12m + 7l + 7m = 0$$

$$45l + 25m = 0$$

$$\frac{l}{m} = \frac{-25}{45} = \frac{-5}{9}$$

Thus R divides PQ externally in the ratio 5: 9

Example: 9

Prove that the three points (2, 5), (4, 6) and (8, 8) are collinear. Find the equation of the line joining them.

Solution :

Let the points be donated by A (2,5), B(4, 6) and C(8,8)

Equation of the line AB is

$$\boldsymbol{y- y_1 = \frac{y_1 - y_2}{x_1 - x_2}\left(x - x_1\right)}$$

$$y - 5 = \frac{5-6}{2-4}(x - 2)$$

$$y - 5 = \frac{1}{2}(x - 2)$$

$$2y - 10 = x - 2$$

$$x - 2y + 8 = 0$$

Put $x = 8$, $y = 8$ the equation is satisfied i.e., the point C (8, 8) lies on the line AB.

Thus the three points. A, B and C are collinear and the equation of the line joining them is $x- 2y + 8 = 0$.

Point of Intersection of Two Lines:

Let two lines L_1 and L_2 intersect at a point P (x_1, y_1) i.e., (x_1, y_1) is a common point on the lines L_1 and L_2. Then (x_1, y_1) will satisfy both the equations representing the lines L_1 and L_2.

In other words x_1 and y_1 give the solution to the two equations which represent L_1 and L_2. So in order to find the point of intersection of two lines, we have to solve their equations for x and y.

Example: 10

Find the point of intersection of the lines $5x + 2y = 11$ and $x - 3y = 9$.

Solution:

$$5x + 2y = 11 \quad \text{..... (1)}$$

$$x - 3y = 9 \quad \text{..... (2)}$$

Multiplying (1) by 3, $15x + 6y = 33$

Multiplying (2) by 2, $2x - 6y = 18$

Adding, $17x = 51$

$$x = 51/17$$

$$x = 3$$

Substituting, $x = 3$ in (1),

$$5(3) + 2y = 11$$

$$15 + 2y = 11$$

$$2y = 11 - 15$$

$$2y = -4$$

$$y = -4/2$$

$$y = -2.$$

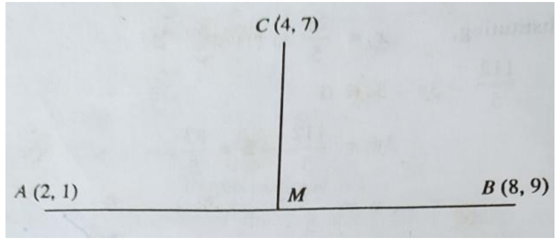
Therefore the point of intersection is (3, -2)

Example 11

Let two cities located at (2, 1) and (8, 9) be connected by a straight road. Let a third city be at (4,7). Find the point on the road which should be connected to the third city so that its distance from the road is least.

Solution:

Let the three cities be donated by A, B, C so that A is the point (2, 1) B(8, 9) and C(4, 7).



Let the perpendicular from C meet the road AB at M. Then M is the point on the road which is the nearest to the city C.

Since M is the point of intersection of the lines AB and CM we first find the equation of the two lines AB and CM and then solve them.

Slope of the line AB = $\frac{y_2 - y_1}{x_2 - x_1}$

Slope of the line AB = $\frac{9-1}{8-2}$

$= \frac{8}{6}$

$= \frac{4}{3}$

The equation of the line is, $\boldsymbol{y - y_1 = m\,(x - x_1)}$

Equation of the line AB is,

$$y - 1 = \frac{4}{3}(x - 2)$$

$$4x - 3y - 5 = 0 \qquad \text{............(1)}$$

CM is perpendicular to AB.

(Slope of CM) x (Slope of AB) = -1

i.e., (Slope of CM) x $\frac{4}{3}$ = -1

i.e., Slope of CM $= -\frac{3}{4}$

Equation of CM is

$$y - 7 = -\frac{3}{4}(x - 4)$$

$$3x + 4y - 40 = 0 \qquad \text{.............(2)}$$

Now we solve (1) and (2)

Multiplying (1) by 4,

$$16x - 12y - 20 = 0$$

Multiplying (2) by 3,

$$9x + 12y - 120 = 0$$

Adding $\qquad 25x - 140 = 0$

$$x = \frac{140}{25}$$

$$= \frac{28}{5}$$

Substituting, $\qquad x = \frac{28}{5}$ in (1)

$$\frac{112}{5} - 3y - 5 = 0$$

$$3y = \frac{112}{5} - 5 = \frac{87}{5}$$

$$y = \frac{29}{5}$$

M is $\left(\frac{28}{5}, \frac{29}{5}\right)$

Example 12

Find the equation of the straight line through the intersection of $2x - 3y + 4 = 0$ and $3x + 4y - 5 = 0$ and parallel to $6x - 7y + 8 = 0$.

Solution:

$$2x - 3y + 4 = 0 \qquad \text{.....(1)}$$

$$3x + 4y - 5 = 0 \qquad \text{.....(2)}$$

(1) × 4 gives, $8x - 12y + 16 = 0$

(2) × 3 gives, $9x + 12y - 15 = 0$

Adding $17x + 1 = 0$

$$x = -\frac{1}{17}$$

Substituting in (1)

$$\frac{-2}{17} - 3y + 4 = 0$$

$$3y = 4 - \frac{2}{17}$$

$$= \frac{66}{17}$$

$$y = \frac{22}{17}$$

Thus the point of intersection of the given lines is $\left(-\frac{1}{17}, \frac{22}{17}\right)$

$$6x - 7y + 8 = 0 \qquad \text{......(3)}$$

$$7y = 6x + 8$$

$$y = \frac{6}{7}x + \frac{8}{7}$$

The slope of the lines is $\frac{6}{7}$.

The required line is parallel to line (3) and

hence its slope is also $\frac{6}{7}$

Equation of the required line is

$$\boldsymbol{y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}\left(x - x_1\right)}$$

$$y - \frac{22}{17} = \frac{6}{7}\left(x + \frac{1}{17}\right)$$

$$17y - 22 = \frac{6}{7}(17x + 1)$$

$$119x - 154 = 102x + 6$$

$$102x - 119y + 160 = 0$$

Interpretations

i. Cost – Output:

The variable cost per unit = Variable cost / Number of units made

= Slope of the line

The change in total cost when one more unit is produced is called the **Marginal Cost.**

The **average cost**, which is defined as the total cost over the number of units produced, is not constant.

Example:

The total factory cost (y) of making x units of a product is given by $y = 5x + 300$, and 75 units are made. Find

(i) the fixed cost

(ii) the variable cost

(iii) the total cost

(iv) the variable cost per unit

(v) average cost per unit and

(vi) the marginal cost.

Solution:

(i) Fixed cost is Rs. 300.

(ii) When x units are produced variable cost is $mx = 5x$.

So here the variable cost = Rs. 5× 75

= Rs. 375

(iii) The total cost $y = mx + c$

$$y = 5 \times 75 + 300$$

$$= \text{Rs. } 675.$$

(iv) The variable cost per unit = Slope of the line = Rs. 5

(v) Average cost = Total cost / Number of units

= 675 / 75

= Rs. 9.

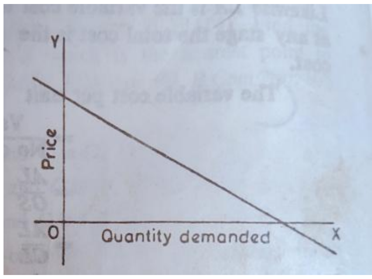
(vi) The marginal cost = Slope of the line = Rs. 5.

Demand and Supply Curves (Linear):

For the sake of simplicity we use linear equations for both demand and supply curves. It is also quite reasonable to do so in certain situations.

Linear Demand Curve:

Usually the slope of a demand curve is negative, that is, as price increases demand decreases demand and as price decreases demand increases.



Example:

15 radios are sold when the price is Rs.400 and 25 radios are sold when the price is Rs.350. What is the equation of the demand curve assuming it to be linear?

Solution:

Denote the demand by x and the price by y.

When, $x_1 = 15$, $y_1 = 400$ and $x_2 = 25$, $y_2 = 350$

Thus the demand curve is the straight line through the points (15,400) and (25,350).

$$\boldsymbol{y} - \boldsymbol{y}_1 = \frac{\boldsymbol{y}_1 - \boldsymbol{y}_2}{\boldsymbol{x}_1 - \boldsymbol{x}_2}\left(\boldsymbol{x} - \boldsymbol{x}_1\right)$$

$$y - 400 = \frac{400 - 350}{15 - 25}(x - 15)$$

$$y - 400 = -5\,(x - 15)$$

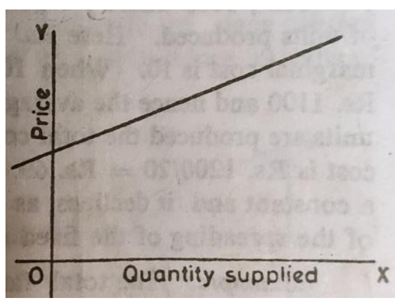
$$y - 400 = -5x + 75$$

$$5x + 75 = 0$$

It is the required equation.

Linear Supply Curve:

Usually the slope of the supply curve is positive as the price increases supply increases and as the price decreases supply decreases.



Example:

When the price is Rs. 50, 60 cameras of a particular type are available and when the price is Rs. 80, 140 cameras of the same type are available in the market. Determine the supply curve.

Solution:

Let x denote the quantity supplied and y the price.

When $x_1 = 60$, $y_1 = 50$ and when $x_2 = 140$, $y_2 = 80$

Thus the supply curve is the straight line through the points (60, 50) and (140, 80).

$$\boldsymbol{y}\text{-}\,\boldsymbol{y}_1 = \frac{\boldsymbol{y}_1 - \boldsymbol{y}_2}{\boldsymbol{x}_1 - \boldsymbol{x}_2}\left(\boldsymbol{x} - \boldsymbol{x}_1\right)$$

$$y - 50 = \frac{50-80}{60-140}(x - 60)$$

$$y - 50 = \frac{3}{8}(x - 60)$$

$$8y - 400 = 3x - 180$$

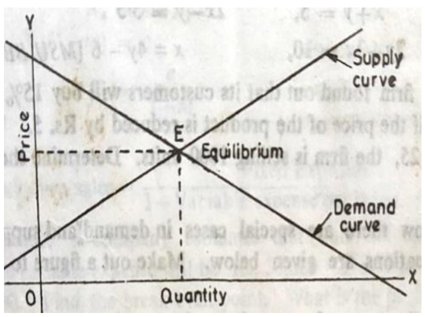
$$3x - 8y + 220 = 0$$

It is the required equation.

Market Equilibrium:

Market equilibrium is said to occur at the point (price) at which the quantity of a commodity demanded is equal to the quantity supplied. Thus the equilibrium amount and equilibrium price correspond to the co-ordinates of the point of intersection of the demand and supply curves.

The equilibrium is meaningless if the point of intersection *E* lies outside the I quadrant.



Applications :

We have considered the case when supply curve and demand curves were straight lines. We take the case when the demand and supply curves are parabolas.

Example

The demand and supply curves are given by

$y = 10 - 3x^2$ and $y = 4 + x^2 + 2x$

(y represents the price and x the quantity).

Find the equilibrium price and quantity.

Solution:

Demand curve , $y = 10 - 3x^2$ ......(1)

Supply curve, $y = 4 + x^2 + 2x$ ......(2)

At the equilibrium level the price for supply and demand are equal. Equating (1) and (2),

$$10 - 3x^2 = 4 + x^2 + 2x$$

$$4x^2 + 2x - 6 = 0$$

$$2x^2 + x - 3 = 0$$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(2)(-3)}}{2 \times 2}$$

$$= \frac{-1 \pm \sqrt{25}}{4}$$

$$= \frac{-1 \pm 5}{4}$$

$$= 1 \text{ or } -\frac{3}{2}$$

$$x = -\frac{3}{2} \text{ is inadmissible.}$$

Substituting x = 1 in (1) we get y = 7. Then at the equilibrium level the quantity supplied and demanded is 1 unit and the price is 7 units.

Break – Even Analysis:

Let us assume that linear relationship exists between the sales revenue and expenses. Let *y* represent the expense incurred when the sales is *x*. Let the relationship be $y = mx + c$.

Let $y = 0.7x + 10{,}800$ …………. (i)

The slope of the line is 0.7. Rs. 0.70 is the variable expense per rupee of sales, for, 0.7 is the change in y for a unit change in x. Rs. 10,800 is the fixed expense.

The **Break Even Point** is the volume of sales at which sales equals expenses, that is, the point at which the company experiences neither a loss nor a profit. Let the break-even level of sales be represented by X_e.

At this stage

$$x = y = X_e \quad \ldots\ldots\ldots\ldots\ldots (2)$$

Using (2) in (1)

$$X_e = 0.7\, X_e + 10{,}800$$

$$0.3\, X_e = 10{,}800$$

$$X_e = 36{,}000$$

Thus when the revenue from sales is Rs. 36,000 the expenses is also Rs. 36,000 and the company has neither profit nor loss.

In general, if the relation is $y = mx + c$,

then the break-even level X_e will be given by

$$X_e = mx + c$$

$$X_e (1 - m) = c$$

$$X_e = \frac{c}{1-m}$$

$$\text{Break} - \text{even sales} = \frac{\text{Fixed Expenses}}{1 - \text{Variable expense per rupee of sales}}$$

Example:

A company estimates that when its sales is Rs. 60,000, its variable expense will be Rs. 30,000 for a fixed expense of Rs. 10,000. Find the break-even point. What is the profit when the sales is Rs. 50,000?

Solution:

The variable expense per rupee of sales = $\frac{Variable\ \ Expense}{Sales}$

$$= \frac{30,000}{60,000}$$

$$= 0.5$$

Now the slope of the line representing the relation between the sales (x) and expenses (y) is 0.5.

$$y = 0.5x + 10,000$$

It is the required equation.

At the break- even level,

$$y = x = X_e$$

$$X_e = 0.5\ X_e + 10,000$$

$$0.5\ X_e = 10,000$$

$$X_e = 20,000$$

This shows that only when the company's sales exceed Rs. 20,000 there will be profit for the company.

When the sales is Rs. 50,000,

$$x = 50,000$$

$$y = mx + c$$

$$y = 0.5 \text{ x } 50,000+10.000$$

$$= 25,000 + 10,000$$

$$= 35,000$$

Profit = Revenue from Sales – Expenses

= Rs. 50,000 – Rs. 35,000

= Rs. 15,000.

Practical Problems:

1) Plot the following points :

A (2,3)	E (0,8)	I (-6,6)
B (-4,-5)	F (3,0)	J (7,-3)
C (2,-4)	G (-2,0)	K (-6,5)
D (-1,3)	H (0,-4)	L (-1,0)
		M (3,3)

2) Take the plan of your city and introduce the axes through an important landmark and locate places of importance using co-ordinates.

3) Referred to some co-ordinate axes an oil well is at the point (3,4) and oil is sent through pipe lines to the port A located at (7,9) and the port B located at (-2,5) (units are in 100 kilometres). Which port is nearer to the site where oil is drilled ?

[Ans.: B is nearer]

4) The plan of an office room is given. The manager's table is at p and one of his assistants is at the table at Q. How far will the office boy have to walk to carry a file from P to Q ? (assume that his path is a straight line). There is a water cooler at R. To whom the water cooler is nearer ?

Ans.: 22.6 m (nearly); To the assistant.]

5) A tourist car was engaged at city A whose co-ordinates are (9,8) to go city B whose co-ordinates are (6,0) and city C co-ordinates (3,-4). Find which is more economical, to go to C first or B first if the charges are the same for return trip also.

[Ans.: A to B and then B to C]

6) Find the slope of the lines:

a) A (2,3) B (-4,-5)	b) A (2,3) D (-1,3)	c) A (2,3) C (2,-4)
d) M (3,3) D (-1,3)	e) I (-6,6) J (7,-3)	f) G (-2,0) E (0,8)
g) K (-6,5) L (-1,0)	h) C (2,-4) M (3,3)	

Ans.: a) 4/3 b) 0 c)$_\infty$ d) 0 e) -9/3 f) 4 g) -1 h) 7

7) Find the equation of the following lines whose slopes and y intercepts are given

a) Slope $-\frac{1}{2}$ y-intercept $\frac{4}{3}$

b) Slope $\frac{2}{5}$ y-intercept 7

c) Slope 3, y-intercept $-\frac{2}{7}$

[Ans.: a) $3x + 6y - 8 = 0$; b) $3x - 5y + 35 = 0$; c) $21x - 7y - 2 = 0$]

8) Find the equation of the line which passes through the given point and having the given slope

a) (5,4) , slope3.

b) (-3,2), slope -2

c) (2,-5) , slope $-\frac{3}{4}$

d) (-4,-6), slope $\frac{3}{2}$

e) (2,-3), slope $\frac{-4}{3}$

[Ans.: a) $3x - y - 11 = 0$; b) $2x + y + 4 = 0$; c) $3x + 4y + 14 = 0$; d) $3x - 24 = 0$; e) $4x + 3y + 1 = 0$]

9) Find the equation of the line which passes through, the pair of points given bellow.

a) (2,4) and (3,4)

b) (-3,1) and (2,-1)

c) (-4,-4) and (5,3)

[Ans.: a) $y = 4$; b) $2x + 5y + 1 = 0$; c) $7x - 9y -8 = 0$]

10) Find the point of intersection of the following lines

a) $x + 3y - 5 = 0$

$x - 2y + 5 = 0$

b) $3x + 4y - 13 = 0$

$2x - 7y + 1 = 0$

c) $x + 3y + 2 = 0$

$2x - y - 3 = 0$

d) $7x - y - 3 = 0$

$2x - 5y - 15 = 0$

[Ans.: a) (-1, 2); b) (3, 1); c) (1, -1); d) (0, -3)]

11) A man has to pay Rs. 5000 initially and then Rs. 300 for every month for 3 years, for a small flat he has bought. Find the relationship between the amount (y) he has paid and the number of months (x) since he bought the flat, assuming the relationship to be linear.

[Ans.: $y = 300x + 5000$]

12) As the number of manufactured units increase from 300 to 600 the total cost of production increases from Rs. 1500 to Rs. 2700. Find the relationship between the cost (y) and the number of units made (x), if the relationship is linear.

[Ans.: $y = 4x + 300$]

13) If taxi fare (y) is Rs. 2 minimum plus 75 paise per kilometre, write the equation connecting the fare and the kilometers travelled (x).

[Ans.: $4y = 3x + 8$]

14) Two cities are located at A (1,2) and B (9,14). Find the point on the road connecting A and B which is the nearest point to the city C located at (6,3).

[Ans.: (3, 5)]

15) solve the previous problem (14) if

i) A is (1, 2), B is (9, 8) and C is (2, 1)

ii) A is (5, 0), B is (-5, 0) and C is (0, 1000)

iii) A is (2, 5), B is (3, 5) and C is (10, 0)

[Ans. i) (29/25, 53/25 ii) (0, 0), iii) (10, 5)]

16) A plant is at A with co-ordinates (0, 2). Its produce is to be trucked to each of the warehouses B and C with co-ordinates (3, 6) and (6, 2) respectively. Find which is more economical: to touch B first or to go to C first if only available roads are between any pair of points. If there were a fourth road connecting B to the nearest point D on the road AC, would it be still better to follow ADBC.

[Ans.: to touch B first is more economical; No]

17) Find the equation of the straight line which makes intercepts a and $3a$ on the co-ordinate axes and passes through the point (3, -4).

[Hint. $(a, 0)$ and $(0, 3a)$ are points on the line]

[Ans.: $3x + y = 5$]

18) If the total cost (y) of making x units is given by

$$y = 4x + 50$$

and if 60 units are made. Find

a) The fixed cost

b) The variable cost

c) The total cost

d) Variable cost per unit

e) Average cost per unit

f) The marginal cost

[Ans.: a)50, b)240, c)290, d)4, e)29/6, f) 4]

19) The total cost (y) of making x units is given by,

$y = 2x + 30$ and 50 units are made. Find

a) The fixed cost

b) The variable cost

c) The total cost

d) Variable cost per unit

e) Average cost per unit

f) The marginal cost

[Ans.: a)30, b)100, c)130, d)2, e)13/5, f)2]

20) When the price of an electric heater is Rs. 30, 100 people will buy and when the price is Rs. 25, 130 people will buy. obtain the equation of the demand curve.

[Ans.: $x + 6y = 280$]

21) At a price of Rs. 7 per bottle a company will supply 6000 squash bottles every month and at Rs. 5 per bottle it will supply 4000 bottles. Find the supply curve.

[Ans.: $x - 1000y + 1000 = 0$]

22) Identify which of the following equations can represent a demand curve and which can represent a supply curve. Graph the curves. Determine the equilibrium point from the graph and verify algebraically. (x quantity, y price)

a) $y = 9 - 2x$ $\quad y = \frac{3}{2}x+1$

b) $y = 15-3x$ $\quad x = 2y-3$

c) $x + y = 5,$ $\quad 2x - y = 5.5$

d) $2y + 3x = 10$ $\quad x = 4y - 6$

[Ans.: a) y = 3/2x + ; (16/7, 31/7), b) x = 2y - 3; (27/7, 24/7, c) 2x - y = 5.5, (3.5, 1.5); x = 4y - 6, (2,2])

23) A firm found out that its customers will buy 15% more of its product if the price of the product is reduced by Rs. 5. When the price is Rs. 25, the firm is selling 1000 units. Determine the demand curve.

[Ans.: $x + 30y = 1750$]

24) A company excepts fixed costs to be Rs . 30000 and variable cost to be Rs . 42000 when the sales will be Rs . 60000

a) Write down the equation relating sales and expenses.

b) Find the break- even point.

c) What will be the profit when the sales is Rs . 120000

d) *[Ans.: y = 0.7x + 30,000; b) Rs. 1,00,000; c) Profit of Rs. 6,000]*

25) A company expects fixed costs of Rs. 37500 and variable cost of Rs. 50000 on sales of Rs. 80000

a) Write down the equation relating the cost and sales

b) Find the break-even point.

c) What will be the profit for a sales of Rs. 90000

[Ans.: y = 0.625x + 37,500; b) Rs.1,00,000; c) Loss of Rs. 3750]

26) A company expects that total fixed cost will be Rs. 25000 and that variable cost will be Rs. 75000 on sales of Rs. 125000

a) Find the relation between total cost and sales

b) Find the break-even point

c) What will be the profit for the sales of Rs. 100000

[Ans.: $y = 0.6x + 25{,}000$; b) Rs.62,500; c) Profit of Rs. 15,000]

27) The fixed cost is Rs. 60000 and cost increases by Rs. 2.5 for each Rs. 4 increase in sales. Find the breakeven point.

[Ans.: Rs. 1,60,000]

28) Find the equilibrium price and quantity for the following :

i) Supply curve ; $y = x^2 + 5x + 1$

Demand curve ; $y = 9 - 2x^2$

ii) Supply curve ; $y = 8x + 7$

Demand curve ; $y = 27 - x^2$

iii) Supply curve ; $4y = x^2 + 2x + 5$

Demand curve ; $y = -2x + 20$

[Ans.: i) (7,1); ii) (23, 2); iii) (10, 5)]

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UNIT – II

SET THEORY

Set theory – definition – types – union, intersection, difference, and complement of sets DeMorgan's Law – Venn diagram – simple set applications – Cartesian product.

Basic Concepts:

Set:

A collection of objects is called a set.

Examples:

(i) The board of directors of a company

(ii) All equipment in a firm

(iii) Consumers of a product

(iv) Employees in a firm

(v) All even numbers.

Element of the set:

An object which belongs to or is a member of a set is called an Element of the set.

Capital letters are used to denote sets while small letters are used to denote the elements of the set, unless otherwise stated.

If x is an element of a set P it is symbolically represented by "$x \in P$"

where '$\in$' stands for, the words "belongs to".

$\notin$ denotes "does not belongs to" .

If y is not an element of a set P then we write $y \notin P$.

A set is designated either by listing all the elements within braces { } or by giving the property which the elements should satisfy to belong to the set.

Example :

The set of all odd numbers less than 7, denoted by P, can be given either as

$P = \{1, 3, 5\}$

by listing all the numbers satisfying the condition, or as

P = {*x*:*x* is an odd number less than 7}

Stands for “such that” . Sometimes a bar "/" is used to denote "such that"

{*x* : *x* is an odd number less than 7}is read as the collection or set of all elements *x* such that *x* is an odd number less than 7.

Example:

Suppose Drs. Rajasekar, Madhavan, Ravi, Rajalingam, Jagdish, Samuel, Kanagalakshmi constitute the Board of Directors of Southern Transport Ccompany.

They form a set A which we write as,

A = {Dr. Rajasekar, Dr. Madhavan, Dr. Ravi, Dr. Rajalingam, Dr. Jagdish, Dr. Samuel, Dr. Kanagalakshmi}.

Or

A = {*x*:*x* is a director of the Southern Transport Company},

The latter representation is very convenient when the set has many elements.

For example:

If we take the set of all men in Chennai city who use Colgate toothpaste, it will be quite unwieldy to write the names of all the persons within the braces.

Whereas we can very briefly write the set as,

S = {*x* : *x* is a person living in Chennai city who uses Colgate toothpaste}

Types of sets :

The different types of sets are as follows:

1. **Finite Set:**

A set having finite number of elements is called a finite set.

For Example:

(i) The board of directors of a company

(ii) All equipment in a firm

(iii) Consumers of a product

(iv) Employees in a firm

2. **Infinite Set:**

A set which is not finite is called an infinite set.

All even numbers serve as an example for an infinite set.

3. **Empty Set** or **Null Set** or **Void Set:**

A set which has no elements is called an Empty Set or Null Set or Void Set and is denoted by ϕ.

Some authors use "O" to denote a null set.

Suppose in Oriental Mills Ltd., the pay scale of senior executives is Rs. 2,000-200-3,000-300-4,500. If we define a set P as follows,

P = {x:x is a senior executive in Oriental Mills limited drawing Rs. 1,900 p.m.}

This set will not have any element and therefore is a null set. Thus,

{x:x is a senior executive in Oriental Mills Ltd. drawing Rs. 1,900 p.m} = ϕ.

4. **Universal Set:**

In any analysis of a particular situation the fixed collection of all elements needed for the analysis is defined as the Universal Set denoted by U (or 1).

Then a particular set is specified by referring to the universal set.

Suppose we want to study about a problem connected with certain number of workers of an industry.

Then the collection of all workers in the industry will naturally be the universal set.

5. **Subset:**

A is said to be a Subset of the set ***B*** if every element of ***A*** is also an element of ***B***.

If ***A*** is a subset of ***B***,

we say ***A*** is contained in ***B*** and

write it symbolically as $A \subseteq B$.

" $\subseteq$ " stands for the words "contained in" or "is a subset of".

Example:

(i) Let ***I*** be the set of integers and ***E*** the set of even numbers. Then $\boldsymbol{E \subseteq I}$**.**

(ii) The set of all consumers of a product in Mumbai is a subset of all consumers of the product in India.

Number of Subsets of a given finite set:

Now let us examine the number of subsets which can be formed out of a given finite set having *n* elements.

Consider first the set having no element.

Only one subset can be formed namely ϕ which is the same as the original set.

Now we take a set containing only one element,

say, *a* i.e. {a}.

Two subsets are possible, the null set and the one element set,

i.e. ϕ, {a}.

Thus the number of subsets that can be formed from a set containing one element is

$2 = 2^1$.

Consider a set with two elements, say, the set {a, b}.

The subsets are, ϕ, {a}, {b}, {a, b}.

The number of such subsets is $4 = 2^2$.

Let {a, b, c} can be a set with three elements a, b and c.

The subsets are,

$$\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}$$

Proceeding like this, by induction, we get the result that,

"A set with *n* elements has 2^n, subsets".

This includes the null set and the given set.

6. **Set Equality:**

Two sets A and B are said to be equal if A has the same elements as in B and vice versa.

That is, two sets A and B are said to be equal if and only if $A \subseteq B$ and $B \subseteq A$.

Consider the sets

$A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5\}$.

$A \neq B$ and neither $A \subseteq B$ nor $B \subseteq A$.

Note:

(i) If two sets are unequal one set may or may not be the subset of the other

(ii) A set is trivially a subset of itself.

(iii) ϕ is a subset of any set.

7. Singleton set :

It has only one element

Eg : A = { 1 }

8. Equal set :

Two sets are equal if they have same elements.

Eg : A = {1, 2, 3, 4, 5, 7}

B = {3, 5, 7, 1, 4, 2}

Here entry element of A is present in B also.

9. Equivalent set :

Two sets are equivalent. If they have same number of elements.

Eg : A = {1, 2, 3, 4}

B = {5, 7, 6, 8}

10.Power set :

A set of every possible subset. The family of all subsets of any set of S and it is denoted by 2^s

Eg : Let S = { a, b, c }

Power set = { s, { a, b }, { a, c }, { b, c }, { a }, { b }, { c }}

11.Disjoint sets :

If no elements of A is in B and no elements of B is in A then we say that A and B are disjoint.

Eg : A = { a, b, c, d }

B = { x, y, z }

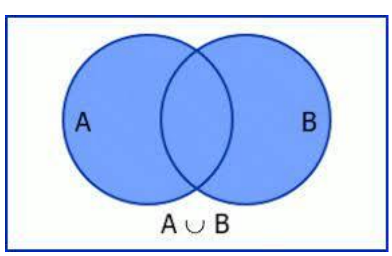
A and B are disjoint sets

UNION:

Let A and B be any two sets. The union of A and B, denoted by A ∪ B, is the set of all elements which are members of either A or B or both.

i.e., A ∪ B = {*x*:*x* € *A* or *x* €*B* or *x* both *A* and *B*}.

The shaded area in figure represents A ∪ B.



Examples:

(i) $A = \{2, 7, 3\}$,

$B = \{4, 5\}$

$A \cup B = \{2, 7, 3, 4, 5\}$

(ii) $A = \{2, 9, 10\}$,

$B= \{9, 10, 1/5, -5\}$

$A\cup B = \{2, 9, 10, 1/5, -5\}$

(iii) $A = \{x, y\}$,

$B = \{y\}$

$A \cup B \{x, y\} = A$

Notice that

$B \subseteq A$. Infact, $A \cup B = A$ if $B \subseteq A$.

(iv) A = {Kannan, Shanmugaraj, Balaji, Balamurugan},

B = {Balaji, Dhakshana, Sujay, Sutharshana, Karunakaran}

A ∪ B ={Kannan, Shanmugaraj, Balaji, Balamurugan, Dhakshana, Sujay, Sutharshana, Karunakaran}

Facts:

(i) For any set A,

$A \cup A = A$

(ii) $A \cup U = U$ and

$A \cup \phi = A$

INTERSECTION:

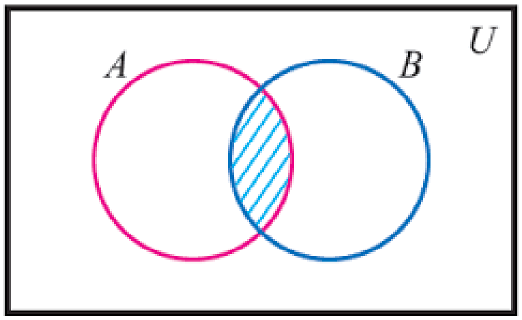
Let A and B be any two sets.

Then the intersection of A and B, denoted by A ∩ B, is the set of all elements which are members of both A and B.

i.e., $A \cap B = \{x : x \in A \text{ and } B\ x \in B\}$.

A ∩ B gives the collection of all elements common to both A and B.

The shaded area in figure represents A ∩ B.



In some books A ∩ B is denoted by AB.

The intersection of A and B, is also called the product of A and B.

Examples:

(i) A = {1, 7, 3},

B = {2, 3, 5}

A ∩ B = {3}

(ii) A = {Kannan, Shanmugaraj, Balaji, Balamurugan},

B = {Balaji, Dhakshana, Sujay, Sutharshana, Karunakaran}

A ∩ B = {Balaji}

(iii) A ={-1, 3, -5, 8},

B = {6, 7, -2, -3}

A ∩ B = ϕ

Two non-empty sets A and B are called disjoint if A ∩ B = ϕ

Facts:

(i) $A \cap A = A$ and $A \cap A = A$

(ii) If $B \subseteq A$, then $A \cap B = B$

(iii) $A \cap \phi = \phi$

Set Difference.

Let A and B be any two sets (subset of some universal set U).

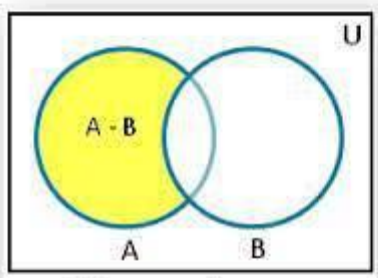
The difference A - B, of two sets A and B is the set whose elements are those elements in A which are not in B

i.e., $A - B = \{x : x \in A \text{ and } x \notin B\}$.

A - B is read as A minus B or A difference B.

A - B is also known as the relative complement of B in A.

The shaded portion in figure represents A - B.



Difference of two sets
A - B

Example:

Let $\cup = \{1, 2, 3, 4\}$

$A = \{1, 2, 3\}$

$B = \{2, 3, 4\}$

Then $A - B = \{1\}$ and

$B - A = \{4\}$

Now, $A \cup B = \{1, 2, 3, 4\}$ and

$A \cap B = \{2, 3\}$

Then $(A \cup B) - (A \cap B) = \{1, 4\}$ ………………….(1)

$A \cap B' = \{1\} = A - B$

This result,

$A - B = A \cap B'$ is always true ……………………..(2)

Similarly $B - A = B \cap A'$

Now $(A - B) \cup (B - A) = \{1, 4\}$ ……………………..(3)

Comparing (1) and (3) we get,

$(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

Using the results (2) and (3) we get,

$(A = B) \cup (B - A) = (A \cap B') \cup (B \cap A')$

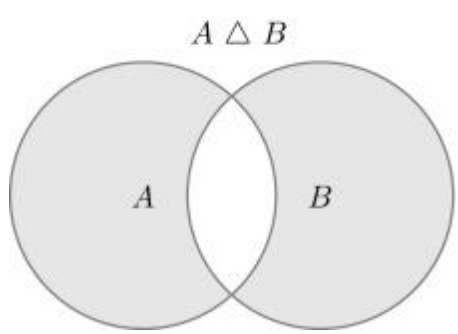
Definition:

The symmetric difference of two sets A and B, denoted by $A \triangle B$ is defined as

$(A \cup B) - (A \cap B)$

This set $(A - B) \cup (B - A)$ is clearly the symmetric difference of the sets A and B.

The shaded area in figure represents $A \triangle B$.



An Important Note:

The operations $\cup$ and $\cap$ are logically equivalent to "or" (inclusive or) and "and" respectively.

Example:

In a market survey people are to be classified according to whether they smoke cigarettes, pipes, cigars or do not smoke all. Let A, B, C be the set of persons who smoke cigarettes, pipes and cigars respectively.

Describe the persons who belong to the following sets.

a) $A \cap B$

b) $A \cap C$

c) $B \cap C$

d) $A \cup B \cup C$

e) $A \cap B \cap C$

f) $A' \cap B' \cap C'$

g) $A' \cup B' \cup C'$

Solution:

(a) Since $A \cap B = \{x : x \in A \text{ and } B\}$,

$A \cap B$ is the set of all persons who smoke both cigarettes and pipes.

(b) Since $A \cap C = \{x : x \in A \text{ and } C\}$,

$A \cap C$ is the set of all persons who smoke both cigarettes and cigars.

(c) Since $B \cap C = \{x : x \in B \text{ and } C\}$,

$B \cap C$ is the set of all persons who smoke both pipes and cigars.

(d) $A \cup B \cup C$ is the set of all persons who smoke cigarettes, or pipes or cigars or any two of them or all the three

i.e., the set of persons who smoke at least one brand.

(e) $A \cap B \cap C$ is the set of all persons who belong to the sets A, B and C, and, therefore the set represents the set of persons who smoke all the three

i.e., cigarettes, pipes and cigars.

(f) $A' \cap B' \cap C'$

By De Morgan's Law,

$$(A \cup B \cup C)' = A' \cap B' \cap C'$$

$A' \cap B' \cap C'$ is the complement of the set $A \cup B \cup C$

i.e., $A' \cap B' \cap C'$ is the set of all persons who do not belong to $A \cup B \cup C$)

i.e., the set of all persons who do not smoke.

(g) $A' \cup B' \cup C'$ is the set of all persons who do not belong to all the three sets A, B, C

i.e., the set of persons who do not smoke all the three varieties.

This is the set of persons who do not smoke at all or who smoke any one only or two at the most

i.e., the set of all persons who smoke at the most two of the three viz. Cigarettes, pies and cigars.

Example :

A market research group conducted a survey of 1000 consumers and reported that 730 consumers liked product A and 455 consumers liked product B. What is the least number that must have liked both products assuming that there may be consumers of products different from A and B.

Solution:

Let A represent the set of all consumers of the product A and

B the set of all consumers of the product of B.

$A \cap B$ gives the set of consumers who use both A and B.

We have,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$\therefore \quad n(A \cap B) = n(A) + n(B) - n(A \cup B)$$

$$= 730 + 455 - n(A \cup B)$$

$$= 1185 - n(A \cup B).$$

When n $(A \cup B)$ is maximum, n $(A \cap B)$ will be minimum.

If all the consumers liked A or B, n $(A \cup B)$ will be 1000 and this is the greatest value possible for n $(A \cup B)$.

$\therefore$ The least value of n $(A \cap B) = 1185 - 1000$

$= 185$

Thus 185 is the least number of people who liked both the products A and B .

Example :

In a survey concerning the smoking habits of consumers it was found that, 55% smoke cigarette A, 50% smoke B. 42% smoke C, 28% smoke A and B, 20% smoke A and C, 12% smoke B and C and 10% smoke all the three cigarettes.

i) What percentage do not smoke?

ii) What percentage smoke exactly two brands of cigarettes?

Solution:

Let A, B, C represent the set of all persons who smoke the brands A, B, C respectively.

i) $A \cup B \cup C$ is the set of all persons who smoke either A or B or C ; or any two brands or all the brands.

Therefore $(A \cup B \cup C)'$ will give the set of all people who do not smoke any one or any two or all the three brands

i.e., the set of people who do not smoke.

Now,

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

$$= 55 + 50 + 42 - 28 - 20 - 12 + 10$$

$$= 97$$

$$n(A \cup B \cup C) + n\{(A \cup B \cup C)'\} = 100$$

$n\{(A \cup B \cup C)'\} = 100 - n(A \cup B \cup C)$

$= 100 - 97$

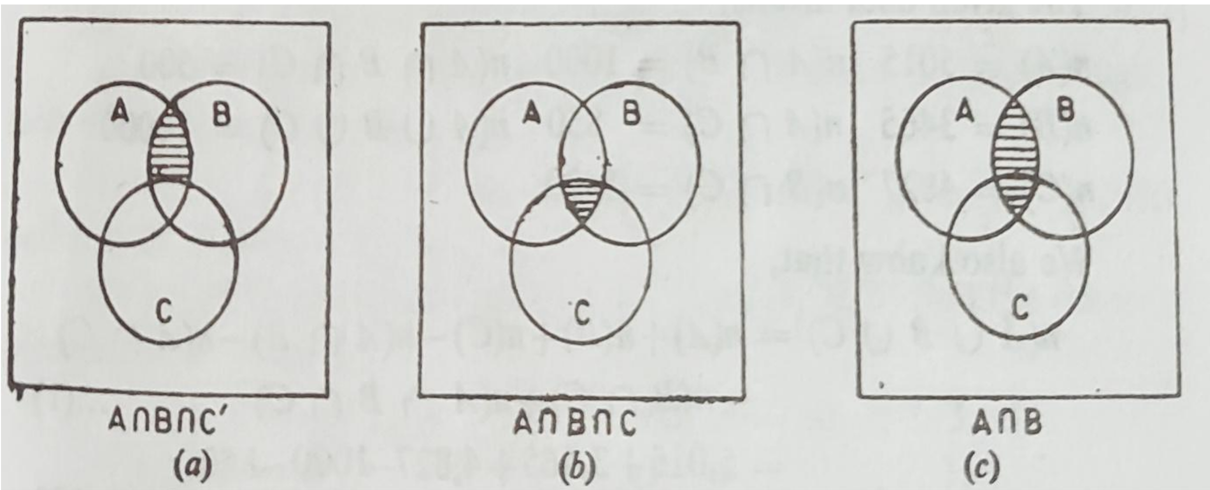
$= 3$

3% do not smoke.

ii) $A \cap B \cap C'$ is the set of persons who smoke A and B but not C.

$A \cap B \cap C$ is the set of persons who smoke all the three brands A, B, C.

The above sets are disjoint and their union is $A \cap B$.



$$n(A \cap B \cap C') + n(A \cap B \cap C) = n(A \cap B)$$

$$n(A \cap B \cap C') + 10 = 28$$

$$n(A \cap B \cap C') = 28 - 10$$

$$n(A \cap B \cap C') = 18$$

Similarly

$$n(A \cap C \cap B') = 20 - 10$$

$$= 10$$

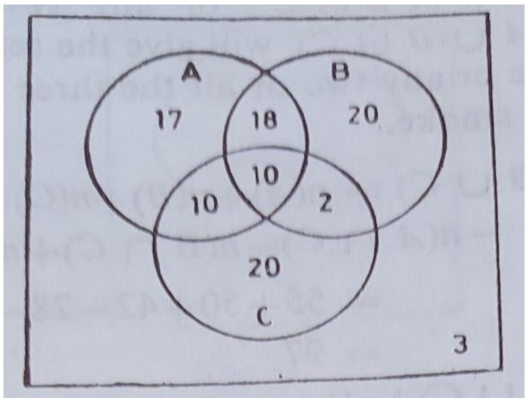
and $n(B \cap C \cap A') = 12 - 10$

$= 2$

$\therefore$ The required number = 18 + 10 + 2

$= 30$

The same fact can be easily seen by drawing the following diagram.



Example :

A company study of the product preferences of 10,000 consumers reported that each of the products A, B, C was liked by 5015; 3465; 4827 respectively and all the products were liked by 500 people, produces A and B were liked by 1000, products A and C were liked by 850 and products B and C were liked by 850 and products A and C were liked by 1,420. Prove that the study results are not correct. It was found that an error was made in recording the number consumers liking the products A and C. What is the value of this numbers?

Solution:

Let A, B, C denote the set of people who like products A, B, C respectively.

The given data means

$n(A) = 5015$

$n(A \cap B) = 1000$

$n(A \cap B \cap C) = 500$

$n(B) = 3465$

$n(A \cap B) = 850$

$n(A \cup B \cup C) = 10000$

$n(C) = 4827$

$n(B \cap C) = 1420$

We also know that,

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap BC) \ldots (1)$$

$$= 5015 + 3465 + 4827 - 1000 - 850 - 1420 + 500$$

i.e., $n(A \cup B \cup C) = 10537 \neq 10000$

This shows that there is an error.

According to the problem there is an error in recording the number $n(A \cap C)$.

Now let us find the correct value of $n(A \cap C)$.

Ignoring the given value of $n(A \cap C)$ and using the result (1)

$$10000 = 5015 + 3465 + 4827 - 1000 - 1420 - n(A \cap C) + 500$$

$$= 13807 - 2420 - n(A \cap C)$$

$$n(A \cap C) = 11387 - 10{,}000$$

$$= 1{,}387$$

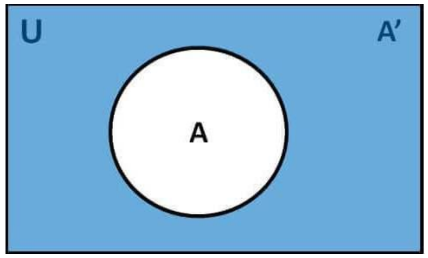
Complement:

Let A be any subset of some universal set U.

Then the complement of A denoted by A' is the set of all elements of U which do not belong to A.

i.e., $A' = \{x : x \in U \text{ and } x \notin A\}$

The shaded portion in figure represents A'.



Some authors use Ā or ~ A to denote the complement.

Examples:

(i) U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10},

A = {3, 5, 9, 10}

Then A' = {1, 2, 4, 6, 7, 8}

(ii) Let N be the set of all positive integers and E, the set of all positive even numbers. Then E' is the set of all positive odd numbers.

(iii) Let U = {*x*:*x* is an administrative officer in a firm} and

A = {Mr. Subramanian, Mr. Kalimuthu, Mr. Kanagappan} and

A ⊆ U.

Then A' = {*x*:*x* ∈ U but *x* is not Mr. Subramanian, Mr. Kalimuthu, Mr. Kanagappan}

(iv) If U = {*x*:*x* is a manufacturing company registered in Tamil Nadu} and

A = {*x*:*x* is a manufacturing company registered in Tamil Nadu producing items and export}

Then A' = {*x*:*x* is a manufacturing company registered in Tamil Nadu producing items for home consumption only}

Note that,

$A \cup A' = U$,

$A \cap A' = \phi$,

$(A')' = A$

De Morgan’s Laws:

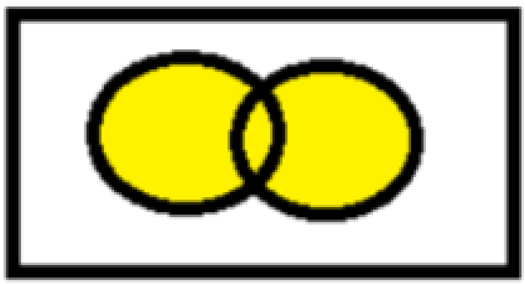
Verify that,

(i) (A ∪ B)’ = A’ ∩ B’

(ii) (A ∩ B)’ = A’ ∪ B’ using Venn diagram

(i) To verify that

(A ∪ B)’ = A’ ∩ B’



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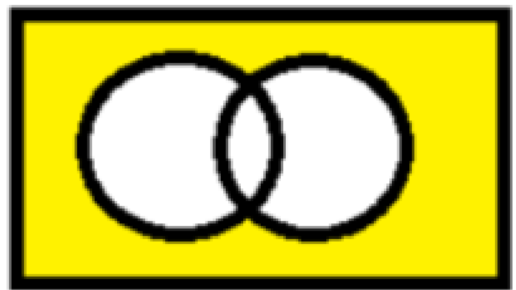
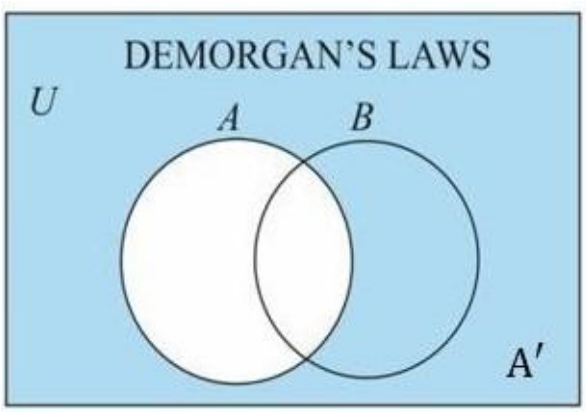
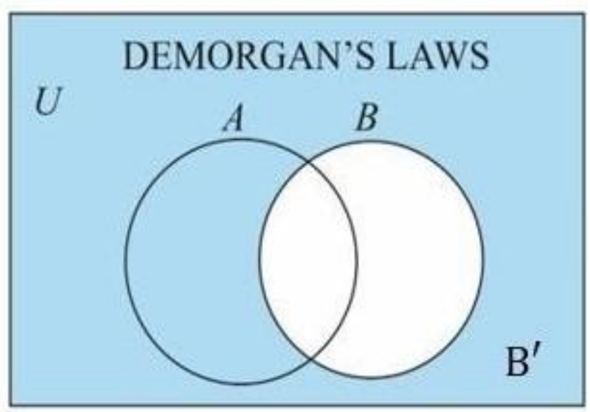
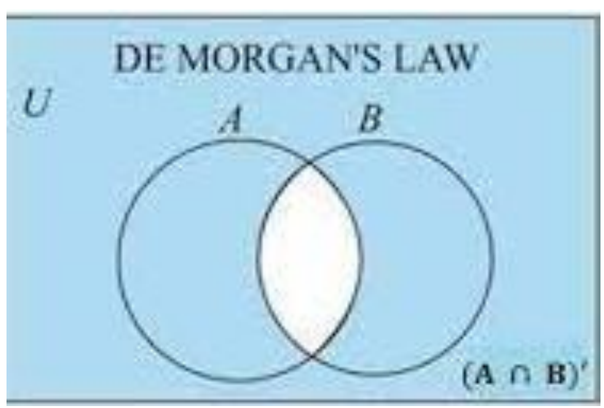


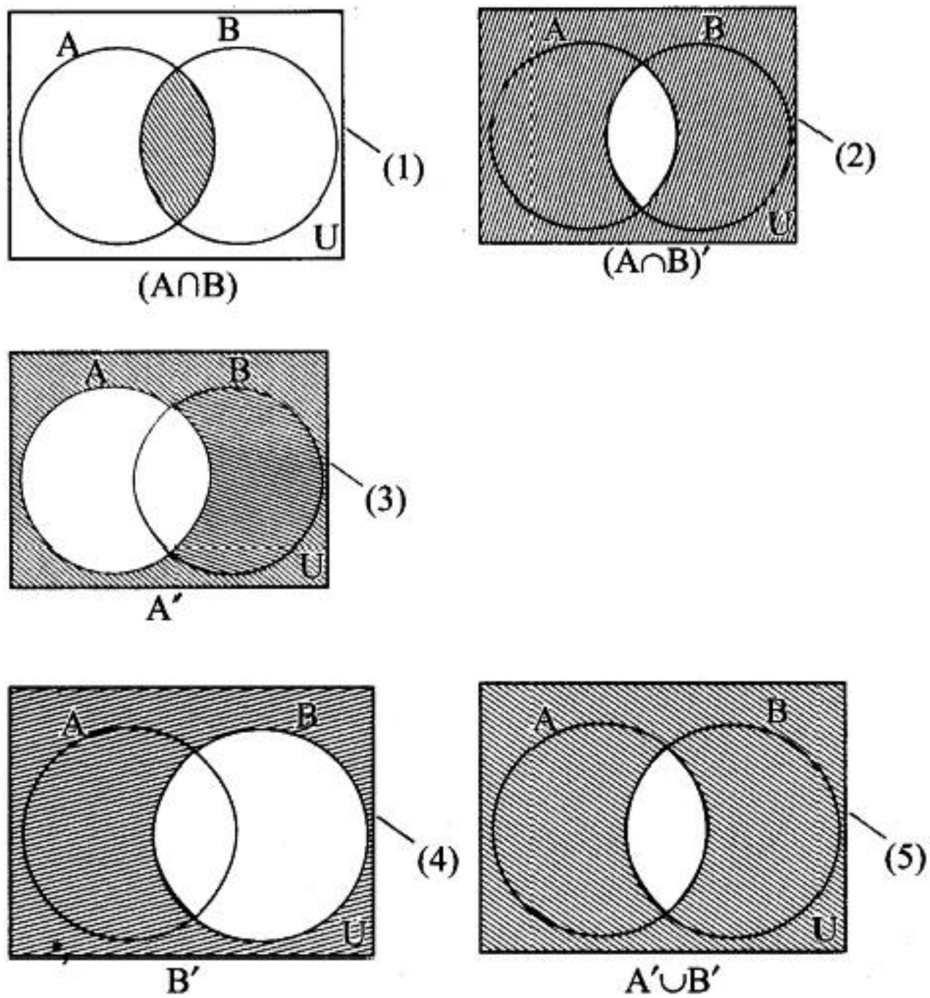
(AUB)'







(i) $(A \cap B)' = A' \cup B'$ using Venn diagram

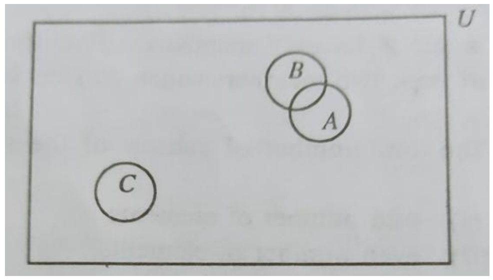


Operations on sets- Applications:

We have seen that one way of generating new sets from the elements of a given universal set is by subset forming. There is another way of forming new sets from given sets using certain operations on sets, which will be given here.

Venn Diagram:

There is a way of visualizing set operations and performing calculations with sets by means of a diagram. A rectangle is used to represent the universal set *U*. All the members of the universal set will be represented by points in it. A circle inside the rectangle will represent a set which is got form *U*, the members of the set being the points inside the circle. To every sub-set of *U* we can associate a circle. Such a diagram is called the Venn Diagram.



Cartesian Product of Two Sets:

Let A and B be any two non empty sets.

Then the Cartesian Product of the two sets A and B denoted by A x B is defined as follows:

A x B $=\{(x, y) : x \in A, y \in B\}$

A x B, read as A cross B is the collection or ordered pairs of elements (x, y)

where x is chosen only from A and y is chosen only from B.

So B x A need not to be the same as A x B.

A x B just gives one more way of producing a new set from two given sets, whose elements are ordered pairs.

We can define A x B x C,

A x B x C x D etc. in the same manner,

For example,

A x B x C $= \{(x, y, z): x \in A, y \in B, z \in C\}$

Example 1:

Let A = {a, b, c, d} B = {1, 2, 3}. Construct the set A x B.

Solution:

	1	2	3
a	(a, 1)	(a, 2)	(a, 3)
b	(b, 1)	(b, 2)	(b, 3)
c	(c, 1)	(c, 3)	(c, 3)
d	(d, 1)	(d, 2)	(d, 3)

A x B = {(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3), (d, 1), (d, 2), (d, 3)}

Note:

(i) A x B contains 12 (= 4 x 3) elements.

(ii) If A and B contain *m* and *n* elements respectively A x B contains *m* x *n* i.e., *mn* elements.

(iii) B x A also contains *mn* elements.

But B x A ≠ A x B if A ≠ B.

Example:

Find A x A where A = {1, 2, 3}

Solution:

	1	2	3
1	(1, 1)	(1, 2)	(1, 3)
2	(2, 1)	(2, 2)	(2, 3)
3	(3, 1)	(3, 2)	(3, 3)

A x A = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}

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Practical Problems:

Problem set 2 (a)

1) Find the subsets of the set A Where $A = \{a, b, c, d, e\}$.
2) Find the total number of

   (i) two element subsets

   (ii) three element subsets of the set $\{x, y, z, w, t\}$
3) Let a set A have 15 members. Find the total number of subsets having at least two members which can be formed out of A.
4) Find the total number of subsets of the set {1, 2, 3, 4, 5} which contain
   - i) Odd number of elements
   - ii) Even number of elements
   - iii) Odd numbers only
   - iv) Even numbers only
5) Consider the set E of all employees in a firm and the subset of persons described by the following definition :

   two persons $x, y \in E$ are in the same subset if they do the same type of job.

   What can you say about the different subsets thus formed.

   [No person does more than one job]
6) If subsets are formed out of A in problem (5) by defining that any two persons, $x, y \in E$ are in the same subset if they get the same salary. What can you about the different subsets thus formed ?
7) A company's research budget is Rs. 60,000. It is considering the allocation of this budget to four research projects with the following requirements for each :

Project	Requirement
1	15,000
2	25,000
3	35,000
4	10,000

Find the total costs of the 16 possible subsets of projects and select the subsets that do not exceed in cost the planned budget.

Problem set 2 (b)

I. 1) Find $A \cup B$ if

i) $A = \{2, 4, 5\}$,
$B = \{4, 7, 12, 13, 0\}$

ii) $A = \{a, b, c\}$,
$B = \{x, y, z\}$

iii) $A = \{5, 6, 7, 8\}$,
$B = \{6, 7\}$

iv) A = {Inthumathi, Meena},
B = {Meena, Merlyjose}

v) $A = \{x : x$ is an even number$\}$,
$B = \{y : y$ is an odd number$\}$

vi) $A = \{x : x$ is a rational number$\}$,

vii) $B = \{y : y$ is an integer$\}$

viii) $A = \{x : x$ is a consumer of a product $P\}$,
$B = \{x : x$ is a consumer of a product $Q\}$

2) (a) Find $A \cup B \cup C$ if

i) $A = \{1, 2, 3\}$, $B = \{2, 3, -5, 0\}$ $C = \{-5, \frac{1}{2}, \log 2, \text{II}\}$

ii) A is the set of positive integers
B is the set of negative integers and
C is the set of all rational numbers

iii) $A = \{1, 2, 0\}$ $B = \{2, 0, 3, 4\}$ $C = \{-1, -2, 0, 2\}$

b) (1) Find $A \cap B$ for the sets A and B given in (i) through (vii) of problem I (a). (i)

2) Find $A \cap B \cap C$ for the sets A, B and C given in (i) , (ii) and (iii) of problem I (2) (a).

(c) (1) If the universe set $U = \{0, 1, 2, 3, 4, 5\}$ find the complement of A when

i) $A = \{0, 1, 3\}$ ii) $A = \{0, 1, 2, 3, 4\}$

iii) $A = \emptyset$ iv) $A = \{0, 1, 2, 3, 4, 5\}$

2) What is the complement of the set of integers when the universal set is the set of all real numbers.

3) What is the complement of the set of all male employees in a firm? (Universe set is the set of all employees in the firm.)

4) What is the complement of all skilled labours in an industry taking the universal set as the set of all labourers in the industry.

(5) If $x = \{0, 1, 2, 3, 4, 5\}$ and $A = \{1, 3, 5\}$, $B = \{0, 5, 2, 4\}$.

Verify De Morgan's Laws, by finding $A' \cup B'$, $A \cup B$, $A' \cap B'$ and $(A \cap B)'$

(d) (1) Find $A - B$ and $B - A$ for the sets A and B given in the problem I (1)

(2) Find $A \Delta B$ for the same set A and B given in I (1).

II. (1) In a market survey people are to be classified whether they use the product of type I or type II or type III or none. Let A, B, C be the set of people who use the types, I, II and III respectively. Describe the persons who belong to the following sets.

a) $A \cup B \cup C$ b) $A' \cup B' \cup C'$ c) AB

d) AC e) BC f) ABC

g) $A(B' \cup C')$ h) $(AB') \cup C'$ i) $A'B'C'$

j) ABC' k) $A \cup (BC)'$ where AB denotes $A \cap B$

2) A survey reports that 78% of those interviewed were married, 46% were married men, 12% were married women with no children and 30% were married women with children. Is the report correct ?

3) In a market survey a manufacturer obtained the following data.

Did you use our brand ?	Percentage answering yes
1. April	59
2. May	62

3. June	62
4. April and May	35
5. May and June	33
6. April and June	31
7. Aril, May and June	22

Is this correct ?

4) Draw the Venn diagram for the following data as in example 4. Assuming that every student in the class takes one of the courses find the total number of students in the class.

	(i)	(ii)
Students taking English	28	36
Students taking French	23	23
Students taking German	23	13
Students taking,		
English and French	12	6
English and German	11	11
French and German	8	4
All the three courses	5	1

Comment on the result in (ii)

5) In a survey concerning the reading habits of students it was found that,

60% read magazine A, 50% read magazine B,

50% read magazine C, 30% read A and B,

20% read B and C, 30% read C and A and

10% read all the three

a) What percentage read exactly two magazines ?

b) What percentage do not read any of the three magazines ?

6) In a survey of 100 students, the numbers studying various languages were found to be,

English 28, German 30, French 42,

English and German 8, English and French 10,

German and French 5, and all the three languages 3.

a) How many students did not study and language ?

b) How many students had French as their only language ?

7) A factory inspector examined the defects in hardness, finish and dimensions of an item. After examining 100 items he gave the following report.

All three defects 5, defect in hardness and finish 10, defect in dimensions and finish 8, defect in dimension and hardness 20. Defect in finish 30, in hardness 23, in dimension 50. The inspector was fined. Why ?

8) The report of the inspector of an assembly line showed the following for 100 units,

Item	Defect	No. Of pieces
1.	Strength defect (S)	35
2.	Flexibility defect (F)	40
3.	Radius defect (R)	18
4.	S and F	7
5.	S and R	11
6.	F and R	12
7.	S, F and R	3

The report was returned. Why ?

9) In a survey of 100 families the numbers that read the most recent issues of various magazines were found to be as follows :

Readers Digest	28
Readers Digest and Science Today	8
Science Today	30
Readers Digest and Caravan	10
Caravan	42
Science Today and Caravan	5
All the Three magazines	3

Using set theory, find

a) How many read none of the three magazines ?

b) How many read Caravan as their only magazine ?

c) How many read Science Today if, and only if they read Caravan ?

10) In a survey of 100 families, the number that read recent issues of a certain monthly magazine were found to be :

September only, 18; September but no August, 23; September and July, 8; September 26; July 48; July and August, 8; none of three months, 24. With the help of set theory find,

a) How many read August issue only ?

b) How many read two consecutive issues ?

c) How many read the July issue, if and only if they did not read the August issue ?

d) How many read the September and August issues, but not the July issue.

11) In a survey of 1000 customers the number of people that buy the various grades of coffee seeds were found to be as follows.

A grade only ...180 *A* grade but not *B* grade 230

A grade and *C* grade	...80	*A* grade	260
C grade	...480	*C* grade and *B* grade	80

None of the three grades 240.

a) How many buy *B* grade coffee seeds only ?

b) How many buy *C* grade, if and only if they do not buy *B* grade ? and

c) How many buy the *C* and *B* grades but not the *A* grade ?

12) The table below records the reaction of a number of listeners to a radio programme. All the categories can be defined in terms of the following four :

M : Males, G : Grown ups

L : Liked slightly V : Liked very much

	Liked very much	Liked slightly	Disliked slightly	Disliked very much
Men	1	3	5	10
Boys	6	5	2	2
Women	6	9	2	1
Girls	8	5	1	1

How many people fall into each

a) M. b) L. c) V. d) $M \cap G' \cap L' \cap V$

e) $M' \cap G \cap L$ f) $(M \cap G) \cup (L \cap V)$

g) $(M \cap G)'$ h) $(M' \cup G')$

i) $M - G$ j) $(M - (G \cap L \cap V')$

13) The workers in a factory were classified according to skill, number of years of service in the factory and whether they performed direct or indirect labour. If they had less than three years of service they were considered as short- term workers; if they served 10 years or more they were

considered long-term workers, and all others were classified medium term workers. Consider the following data.

	Skilled and direct	Unskilled and direct	Skilled and indirect	Unskilled and indirect
Short term	6	8	10	20
Medium term	7	10	16	9
Long term	3	2	8	0

If *S*, *M*, *L*, *SK*, *I* denote short, medium long term and skilled and indirect respectively,

i) Determine the number of workers in the following classes

a) M. b) $L \cap I$, c) $S \cap SK \cap I$

d) $(M \cup L) \cap (SK \cup I)$ e) $S' \cup (S' \cap I)'$.

ii) Which set of the following pairs has more workers as its members.

a) $(S \cup M)'$ or L

b) $I \cap (SK)'$ or $S - (I \cap S')$.

14) A survey reports that 80% of the population read a magazine 'M' and 55% are males. Find the least possible percentages of males reading 'M' and of females reading M.

15) A market research group conducted a survey of 1000 consumers and reported that 720 consumers liked product *A* and 450 consumers liked product *B*. What is the least number that must have liked both products ?

16) In a town 60% read magazine A, 25% do not read magazine A but read magazine B. Calculate the percentage of those who do not read any. Also find the highest and lowest possible figures of those who read magazine B.

Problem set 2 (c)

1) Find A×B (i) if $A = \{1, 2, 3\}$ and $B = \{2, 4\}$

(ii) if $A = \{a, b, c\}$ and $B = \{x, y, z\}$

(iii) if A = {x : x is a real number}= B

Comment on the result you have obtained in (iii) and extend it to $A \times A \times A$

2) Find $A \times B \times C$ where

(i) $A = \{1, 2\}$, $B = \{0, 5\}$ and $C = \{6, 7\}$

(ii) $A = \{1, 2, 5\}$, $B = \{2, -1\}$, $C = \{0\}$

(iii) $A = \{7, 8, 9\}$, $B = \{0, 10\}$, $C = \emptyset$

Answers:

Problem set 2 (a)

1. ∅, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {a, e}, {b, c}, {b. d}, {b, e}, {c, d}, {c, e}, {d, e}, {a, b, c}, {a, b, d}, {a, b, e}, {a, c, d}, {a, c, e}, {a, d, e}, {b, c, d}, {b, c, e}, {b, d, e}, {c, d, e}, {a, b, c, d}, {a, b, c, e}, {a, b, d, e}, {a, c, d, e}, {b, c, d, e}, {a, b, c, d, e}.

2. *10 , 10*

3. $2^{15} - 16$ = *32752*

4. (i) 16

(ii) 16

(iii) 7

(iv) 3

5. No two subsets have elements in common

6. Same as (5)

7. ∅ , 0 {2, 3}, 60,000

{1}, 15,000 {2, 4}, 35,000

{2}, 25,000 {3, 4}, 45,000

{3}, 35,000 {1, 2, 3}, 75,000

{4}, 10,000   {1, 2, 4}, 50,000

{1, 2}, 40,000   {1, 3, 4}, 60,000

{1, 3}, 50,000   {2, 3, 4}, 70,000

{1, 4}, 25,000   {1, 2, 3, 4}, 85,000

Except the projects {1, 2, 3}, {2, 3, 4} and {1, 2, 3, 4} all the rest do not exceed the budget.

Problem set 2 (b)

I. 1. (i) {2, 4, 5, 7, 12, 13, 0}

(ii) {a, b, c, x, y, z}

(iii) {5, 6, 7, 8}

(iv) {Inthumathi, Meena, Merlyjose}

(v) { x : x is an integer}

(vi) { x : x is a rational number }

(vii) { x : x is a consumer of the product P or the product Q or both }

2. (a) (i) $\{1, 2, 3, -5, 0, \frac{1}{2}, \log 2, \pi\}$

(ii) $\underline{C}$

(iii) {1, 2, 0, 3, 4, -1, -2}

(b) 1. (i) {4}

(ii) $\emptyset$

(iii) {6, 7}

(iv) {Meena}

(v) $\emptyset$

(vi) { x : x is an integer}

(vii) { x : x is a consumer of both the products P and Q}

2. (i) ∅

(ii) ∅

(iii) {2, 0}

(c) 1. (i) {2, 4, 5}

(ii) {5}

(iii) {0, 1, 2, 3, 4, 5} = U (iv) ∅

2. The set of all non- integral real numbers.

3. The set of all female employees in the firm.

4. The set of all unskilled labourers.

(d) 1. (i) {2,5}, {7, 12, 13, 0}

(ii) A , B

(iii) {5, 8} : ∅

(iv) {Jones, Mohammed}; {Paul, Sultan}

(v) A ; B

(vi) The set of all non-integral real numbers; ∅

(vii) { x : x is a consumer of the product P alone};

{ x : x is a consumer of the product Q alone}

2. (i) {2, 5, 7, 12, 13, 0}

(ii) { a, b, c, x, y, z}

(iii) {5, 8}

(iv) { Jones , Mohammed, Paul, Sultan}

(v) { The set of all integers}

(vi) { The set of all non- integral real numbers}

(vii) { x : x is a consumer of P alone or Q alone but not a consumer of both the products}

II.1. (a) The set of all persons who use the product I or II or III or any two of them or all the three.

(b) The set of all persons who do not use all the three.

(c) The set of all persons who use the product of type I and II.

(d) & (e) similar to (c).

(f) The set of all persons who use all the three products I, II and III,

(g) The set of all persons who use products I and II but not III, I and III but not II or I alone.

(h) The set of all persons who use I and III only or the persons who do not use III.

(i) The set of all persons who do not use any of the products.

(j) The set of all persons who use I and II but not III.

(k) The set of all persons who use I or who use II without using III.

2. No.

3. No.

4. (i) 48, (i) The data given is not correct

5. (a) 50%

(b) 10%

6. (a) 20%

(b) 30%

7. The number of items with defect in hardness alone is -2, which is impossible.

8. The number of items with radius defect alone is -2, which is impossible.

9. (a) 20

(b) 30

(c) 5.

10. (a) 10

(b) 8

(c) 40

(d) none.

11. (a) 1000

(b) 400

(c) 50

12. (a) 34

(b) 22

(c) 21

(d) 6

(e) 9

(f) 19

(g) 48

(h) 48

(i) 15

(j) 22.

13. (i) (a) 42

(b) 8

(c) 10

(d) 43

(e) 99.

(ii) (a) Both have equal number of workers.

(b) S – (I ∩ S'), has more workers.

14. 35%, 25%

15. 1170.

16.15%, 85%, 25%.

Problem set 2 (c)

1. (i) {(1, 2), (1,4), (2, 2), (2, 4), (3, 2), (3, 4)}.

(ii) {(a, x), (a, y), (a, z), (b, x), (b, y), (b, z), (c, x), (c, y), (c, z)}.

(iii) Each element corresponds to the x and y co- ordinates of a point in the plane and conversely. Every element in A×A×A corresponds to a point in space and conversely.

2. (i) {(1, 0, 6), (1, 0, 7), (2, 0, 6), (2, 0, 7), (1, 5, 6), (1, 5, 7), (2, 5, 6), (2, 5, 7)}.

(ii) {(1, 2, 0), (1, -1, 0), (2, 2, 0), (2, -1, 0), (5, 2, 0), (5, -1, 0)}

(ii) ∅.

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UNIT- III

DIFFERENTIAL CALCULUS

Differential calculus – derivative of a function – differentiation – standard forms – sum, product, quotient rule – differential coefficients of simple functions (trigonometric functions excluded) – function of a function rule – simple application to business using marginal concept.

Calculus - Introduction

Sir Isacc Newton of England and Gottfried Wilhelm Leibnitz of Germany invented calculus in the 17^{th} century, independently.

Leibniz's calculus originated from his attempts to solve some problems in geometry. Newton's calculus originated from his attempts to solve some problems in physics and astronomy.

Calculus - Origin

'Calculus' is a Latin word which means a 'pebble' or a 'small stone'. In ancient times, pebbles were used as for calculations.

Calculus - Meaning

Calculus is the mathematics of motion and change. When increasing or decreasing quantities are made the subject of mathematical investigation, it frequently becomes necessary to estimate their rate of growth or decay.

Basic Operations of Calculus

The basic operations of calculus are divided into two main parts. They are:

I. Differential Calculus and
II. Integral calculus

DIFFERENTIAL CALCULUS

Differential calculus deals with the rate of change of one quantity with respect to another.

OBJECTIVES OF THE DIFFERENTIAL CALCULUS

The main objective of the differential calculus is to describe an instrument for the measurement of such rates to frame rules for its formation and use.

USES OF DIFFERENTIAL CALCULUS

i) It is used in calculating the rate of change of velocity of a vehicle with respect to time.

ii) It is used in calculating the rate of change of growth of population with respect to time.

iii) Differential calculus is also a method by which maxima and minima of functions are obtained.

iv) Marginal analysis is one of the important applications of calculus in business and economics.

v) The problems of maximizing profit and minimizing cost under various assumptions can be solved.

Differential Coefficient:

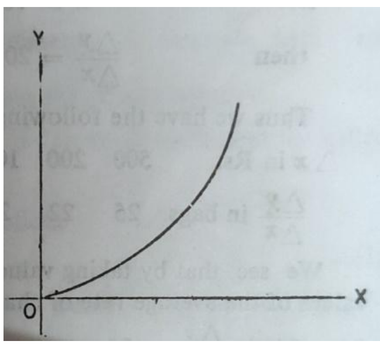
Consider the function,

$$f(x) = \frac{x^2}{100}, x \geq 0$$

Where x is the investment in rupees and

$y = f(x)$ is the total production of paddy in bags.

The empirically meaningful part of the graph of the given function is shown in Fig.



Now the problem is what, is the marginal rate of paddy output at $x = 1000$ (Rupees).

We cannot use $\frac{\Delta y}{\Delta x}$ over a given increment Δx to find the marginal rate at $x = 1000$, since marginal rate is change for any Δx may be.

However we can approximate this marginal rate by computing $\frac{\Delta y}{\Delta x}$ for smaller and smaller values of Δx.

Let us consider the average rate of paddy out-put over a flexible increment from $x = 1000$ to $x = 1000 + \Delta x$.

$$f(1000) = \frac{1000^2}{100} = 10{,}000$$

$$f(1000 + \Delta x) = \frac{(1000 + \Delta x)^2}{100}$$

$$= 10{,}000 + 20\Delta x + \frac{(\Delta x)^2}{100}$$

$$\Delta y = f(1000 + \Delta x) - f\ (1000)$$

$$= 20\Delta x + \frac{(\Delta x)^2}{100}$$

$$\frac{\Delta y}{\Delta x} = 20 + \frac{\Delta x}{100}, \qquad \Delta x \neq 0.$$

Here $\frac{\Delta y}{\Delta x}$ gives the average rate of paddy out-put over an incremental investment

Δx beginning at $x = 1000$ and

ending at $x = 1000 + \Delta x$.

This increment is flexible in the sense that we can make its value equal to any quantity we wish by setting Δx equal to that quantity.

Let $\Delta x = 500$,

then $\frac{\Delta y}{\Delta x} = 20 + \frac{500}{100}$

$= 25$

Let $\Delta x = 200.$

then $\frac{\Delta y}{\Delta x} = 20 + \frac{200}{100}$

$= 22$

Let $x = 100$

then $\frac{\Delta y}{\Delta x} = 20 + \frac{100}{100}$

= 21.

Thus we have the following table,

Δ x in Rs.	500	200	100	50	10	0.1	0.01
$\frac{\Delta y}{\Delta x}$ in bags	25	22	21	20.5	20.1	20.01	20.0001

We see that by taking values by Δx near enough to zero we get values of the average rate of change as close to 20 bags as we wish.

So $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 20.$

20 bags is the exact marginal rate at $x = 1000.$

The above process is called the differentiation of the function $y = \frac{x^2}{100}$ with respect to the independent variable x at $x = 1000.$

Different ial Coefficient or Derivative of y with respect to x

Definition:

Consider a function $y = f(x).$

Let Δx be the increment given to x and

let Δy be the corresponding increment in $y.$

Then the limit of the ratio $\frac{\Delta y}{\Delta x}$ as Δx tends to zero, if it exists, is called the $Differential\ Coefficient\ or\ Derivative\ of\ y\ with\ respect\ to\ x.$

The limit is denoted symbolically as $\frac{dy}{dx}$.

i.e., $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

In other words, if $y = f(x)$ then

$$y + \Delta y = f(x + \Delta x)$$

$$\Delta y = (y + \Delta y) - y$$

$$= f(x + \Delta x) - f(x)$$

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Note:

1.Various other notations for $\frac{dy}{dx}$: $f'(x) or\ y_1\ or\ y'\ or\ Dy\ or\ D\ f\ (x)$

$$where\ D = \frac{d}{dx}\ and$$

$$y = f\ (x).$$

2. $\frac{d}{dx}$stands for the statement “differentiate with respect to x" just like √ means taking the square root.

3. The process of finding the differential coefficient is called differentiation.

Example

Consider the function,

$$f(x) = x^2 \qquad \ldots.(1)$$

Let Δx be the increment given to x and Δy be the corresponding increment in $y = f(x)$.

Then $f(x + \Delta x) = (x + \Delta x)^2$

$$= x^2 + 2x(\Delta x) + (\Delta x)^2 \qquad \ldots.. (2)$$

(2) – (1) gives,

$$\Delta y = f(x + \Delta x) - f(x)$$

$$= 2x\,(\Delta x) + (\Delta x)^2$$

$$\frac{\Delta y}{\Delta x} = \frac{2x(\Delta x) + (\Delta x)^2}{\Delta x}$$

$$= 2x + (\Delta x)$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x)$$

$$= 2x$$

This quantity $2x$ obtained is the differential coefficient of the function x^2 with respect to x.

$$\therefore \frac{d}{dx}(x^2) = 2x.$$

or

$$\frac{dy}{dx} = 2x \qquad \text{where } y = x^2$$

or

$$f'(x) = 2x \qquad \text{where } f(x) = x^2$$

or

$$D\,x^2 = 2x$$

or

$$Dy = 2x \qquad \text{where } y = x^2$$

or

$$Df\,(x) = 2x \qquad \text{where } f(x) = x^2$$

Here D stands for $\frac{d}{dx}$.

Standard Forms

We give here some standard results.

First we give the various stages in the process of finding the differential coefficient of a function using the definition of the differential coefficient alone.

Unless otherwise stated all the functions we consider will be continuous functions only.

The various stages are,

i. Give an increment Δx to the independent variable x.

ii. Find the corresponding increment Δy in y

where $y = f(x)$.

iii. Find $\frac{\Delta y}{\Delta x}$(the average rate of change).

iv. Evaluate $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ (the marginal or instantaneous rate of change).

Standard Form I

$$\frac{d}{dx}(x^n) = nx^{n-1}, \text{ where } n \text{ is any constant .}$$

The proof of this result is not difficult

For simplicity let us consider,

$$y = x^n$$

Where n is any positive integer.

Then $y + \Delta y = (x + \Delta x)^n$

$\therefore\ \Delta y = (x + \Delta x)^n - x^n$

Using Binomial theorem,

$(x + \Delta x)^n = x^n + nx^{n-2}\Delta x + \frac{n(n-1)}{1.2} x^{n2} (\Delta x)^2 + \ldots\ldots\ldots\ldots\ldots$ terms containing higher powers of Δx.

$\therefore (x + \Delta x)^n - x^n = nx^{n-1}\,\Delta$ + terms containing $(\Delta x)^2$ and higher powers of Δx.

$\frac{(x+\Delta x)^n - x^n}{\Delta x} = nx^{n-1}$ + terms containing (Δx) and higher powers of Δx.

$\frac{\Delta y}{\Delta x} = nx^{n-1}$ + terms containing (Δx) and higher powers of Δx.

$$\therefore \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = nx^{n-1}$$

This result is true when n is any constant and proof can be supplied by using the corresponding binomial theorem.

Power Function Rule

The derivative of a power function: x^n is nx^{n-1}

$$\text{If } y = x^n$$

$$\text{Then, } \frac{dy}{dx} = \frac{dx^n}{dx} = nx^{n-1}$$

Example 1

If $y = x^6$ Find $\frac{dy}{dx}$

Solution:

$\frac{dy}{dx} = \frac{d}{dx}(x^6)$

$= 6x^{6-1}$

$= 6x^5$

Example 2

If $y = x$. Find $\frac{dy}{dx}$

Solution:

$\frac{dy}{dx} = \frac{d}{dx}(x)$

$= 1x^{1-1}$

$= 1x^0$

$= 1$

Standard Form II

Find the differential coefficient of a constant with respect to x.

` Let $y = c$,

where c is any constant.

The problem is to find $\frac{dc}{dx}$

$i.e.$ to find the instantaneous rate of change of a function which remains constant always. Obviously it is zero,

$$i.e. \frac{dc}{dx} = 0.$$

It can be seen in the following way also.

Let $y = c$ …………….(1)

Then $y + \Delta y = c$ ....………….(2)

Since c is constant not depending upon any variable.

(2) – (1) gives,

$$\Delta y = 0$$

$$\therefore \frac{\Delta y}{\Delta x} = 0 \; as \; \Delta x \neq 0$$

$$\therefore \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} 0 = 0.$$

Thus we have,

$\frac{dc}{dx} = 0$, where c is any constant.

Constant Rules

The derivative of a constant is zero

If $y = c$ (constant)

Then, $\frac{dy}{dx} = \frac{d\,c}{dx} = 0$

Example:

If $y = 5$ (constant)

Then, $\frac{dy}{dx} = \frac{d(5)}{dx} = 0$

Some other Important Rules

Sometimes we have to differentiate exponential logarithmic and functions. Their derivatives are:

Exponential function

If $y = e^x$

then $\frac{dy}{dx} = e^x$

Standard Form III

Differential Coefficient of e^x

The function e^x is called the $Exponential\ Function$.

e is an irrational number and

it is a specific number which lies between 2.7 and 2.8.

The function e^x arises in problems involving sales decay, present and future value of an investment in problems of finance, population growth etc.

Once we have a new function one natural question is what is the differential coefficient.

Let, $y = e^x$ .... (1)

and let Δx be the increment given to x and

let $\Delta\, y$ be the corresponding increment in y.

Then $y + \Delta y = e^{x+\Delta x}$ .... (2)

(2) - (1) gives

$$\Delta y = e^{x+\Delta x} - e^x$$

$$= e^x(e^{\Delta x} - 1) \qquad \because a^{m+n} = a^m a^n$$

$$\frac{\Delta y}{\Delta x} = \frac{e^x(e^{\Delta x} - 1)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} e^x \frac{(e^{\Delta x} - 1)}{\Delta x}$$

$$e^x \lim_{\Delta x \to 0} \left[\frac{e^{\Delta x} - 1}{\Delta x}\right] \text{ since } e^x \text{ does not involve } \Delta x$$

$$= e^x \times 1 = e^x$$

$$= e^x \times 1 \qquad \text{using the standard result}$$

$$d(e^x)/dx = e^x \qquad \lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

A Useful Rule in Differentiation

Consider a function $y = f(x)$.

Let Δx be the increment in x and

let Δy be the corresponding increment in y.

Clearly $\frac{\Delta y}{\Delta x} = \frac{1}{\frac{\Delta x}{\Delta y}}$

Taking the limit as $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{1}{\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta y}}$$

$\frac{dy}{dx} = \frac{1}{\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta y}}$, since y is a continuous function of x

$= \frac{1}{\frac{dx}{dy}}$, by definition

Therefore $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

This rule will be every useful whenever we wish to find $\frac{dy}{dx}$ and

finding $\frac{dx}{dy}$ is simpler than finding $\frac{dy}{dx}$.

Logarithmic function

If $y = log\ x$

then $\frac{dy}{dx} = \frac{1}{x}$

Standard Form IV

Differential Coefficient of log e^x:

Let $y = \log e^x$ …… (1)

The problem is to find $\frac{dy}{dx}$.

From (1) $x = e^y$ by the definition of logarithms. …… (2)

$$\therefore \frac{dx}{dy} = e^y \quad \text{…… (3)} \quad \text{(Standard Form)}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$= \frac{1}{e^y}, using\ (3)$$

$$\frac{d(\log x)}{dx} = \frac{1}{x} \text{ (using (2))}$$

Note:

1. In calculus, unless otherwise stated, the base of logarithm is understood to be $'e'$.

2. How to find the differential coefficient of log a^x

when a is not necessarily the same as e ?

We use the change of base rule and standard form IV to find this.

Standard Form V

To find the differential coefficient of a^x with respect to x.

Let $y = a^x$ …. (1)

To find $\frac{dy}{dx}$

Then $x = log_a$ y from the definition of logarithm.

$$x = \frac{log_e\ y}{log_e\ a} \quad \text{…..(2)}$$

Using change of base rule.

Differentiating both sides of (2) with respect to $'y'$

We have,

$$\frac{dy}{dx} = \frac{1}{log_e\ a} \frac{d}{dy} (log_e\ y)$$

$$= \frac{1}{y \, log_e \, a}$$

But $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

$\therefore \frac{dy}{dx} = \frac{1}{\frac{1}{y \, log_e \, a}}$

$\frac{dy}{dx} = a^x log_e \, a \, from$ (1).

RULES OF DIFFERENTIATION

Sum Rule:

Differential coefficient of the sum of two functions,

Let $y = u + v$ ……………. (1)

Where $u \equiv u(x)$, $v \equiv v(x)$ are functions of x.

For an increment $\Delta x \; in \, x$, there will be changes in $u(x)$ and $v(x)$ and also y as y is the sum of u and v.

Let $\Delta u, \Delta v \; and \; \Delta y$ be the corresponding increments.

Then $y + \Delta y = u + \Delta u + v + \Delta v$ ....(2)

(2) - (1) gives

$$\Delta y = \Delta u + \Delta v$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u + \Delta v}{\Delta x}$$

$$= \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left\{ \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} \right\}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}$$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$i,e\ \frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Note:

(1). This can be extended to a sum of any finite number of functions.

$$\frac{d}{dx}(u+v+w) = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx}$$

(2). $\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$

(3). $\frac{d}{dx}(u+v-w) = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}$.

The derivative of a sum of two functions is the sum of the derivatives of the two functions.

If $y = u + v$

Then, $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

Illustration :

Differentiate: $y = 3x^2 + 5x^4$

Solution:

$\frac{dy}{dx} = \frac{d}{dx}(3x^2 + 5x^4) =$

$= \frac{d}{dx}(3x^2) + \frac{d}{dx}(5x^4)$

$= 6x + 20x^3$

Difference Rule:

The derivative of difference of two functions is the difference of the derivatives of the two functions.

If $y = u - v$

Then, $\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$

Illustration :

Differentiate: $y = 2x^2 - 5x^3 + 5$

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}(2x^2 - 5x^3 + 5)$$

$$= \frac{d}{dx}(2x^2) - \frac{d}{dx}(5x^3) + \frac{d}{dx}(5)$$

$$= 4x - 15x^2$$

Product Rule:

The derivative of the product of two functions is equal to the first function times the derivative of the second function plus the second function times the derivative of the first function.

$$y = uv$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Differential coefficient of the product of two functions.

Let, $y = uv$. ……. (1)

Where, $u \equiv u(x), v = v(x)$ be two functions of x.

Let Δx be the increment given to x and let $\Delta u, \Delta v, \Delta y$ be the corresponding increments in u, v, y respectively.

Then $y + \Delta y = (u + \Delta u)(v + \Delta v)$

$$= uv + u\Delta u + v\Delta u + \Delta u\Delta v\Delta \quad \text{……………. (2)}$$

(2) – (1) gives

$$\Delta y = u\Delta v + v\Delta u + \Delta u\Delta v$$

$$\frac{\Delta y}{\Delta x} = u\frac{\Delta u}{\Delta x} + v\frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x}\Delta v$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left\{ u \frac{\Delta v}{\Delta x} + v \frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x} \Delta v \right\}$$

$$= \lim_{\Delta x \to 0} u \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} v \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \Delta v$$

$$= u \frac{dv}{dx} + v \frac{du}{dx}, \text{ Since Limit } \frac{\Delta u}{\Delta x} \Delta v = 0$$

For, $\Delta v \to 0$ as $\Delta x \to 0$since v is a continuous function and

$$\frac{du}{dx} . 0 = 0$$

$$\frac{d}{dx}(uv) = u + \frac{dv}{dx} + v \frac{du}{dx}$$

This rule is called the product rule of differentiation.

This can be extended to the product of any finite number of functions.

Notice that in the right hand side each term contains the differential coefficient of one function and the other function remains the same.

No function is left "undifferentiated".

When $y = uvw \; where \; u, v, w$ are functions of x,

$$\frac{d(uvw)}{dx} = vw \frac{du}{dx} + uw \frac{dv}{dx} + uv \frac{dw}{dx}$$

Note:

Suppose $y = ku \; where \; k$ is a constant and

u is a function of x.

Then, $\frac{dy}{dx} = k \frac{du}{dx} + u \frac{dk}{dx}$

$$= k \frac{du}{dx} \left(\because \frac{dk}{dx} = 0 \right)$$

Thus we have the useful result,

$\frac{d}{dx}(ku) = k \frac{du}{dx}$ where k is a constant.

By this result,

$$\frac{d}{dx}(2x^2) = 2\frac{d}{dx}(x^2) = 2 \times 2x = 4x$$

$$\frac{d}{dx}(-4x^3) = -4\frac{d}{dx}(x^3) = -4 \times 3x^2 = -12x^2.$$

Illustration :

Differentiate: $y = (4x + 8)(4x^3)$

Solution:

$u = 4x + 8$; $v = 4x^3$

$\frac{dy}{dx} = \frac{d(uv)}{dx} = \frac{d}{dx}(4x + 8)(4x^3)$

$= (4x + 8)\frac{d}{dx}(4x^3) + (4x^3)\frac{d}{dx}(4x + 8)$

$= (4x + 8)\, 12x^2 + (4x^3)(4)$

$= 48x^3 + 96x^2 + 16x^3$

$= 64x^3 + 96x^2$

Quotient Rule:

The derivative of the quotient of two functions is given by the following formula.

If u and v are differential function and if $v(x) \neq 0$,

Then, $y = \frac{u}{v}$

$$\frac{dy}{dx} = \frac{\frac{vdu}{dx} - \frac{udv}{dx}}{v^2}$$

Differential coefficient of the quotient of two functions.

Let y be the quotient of two functions $u(x)$ and $v(x)$.

$$i.e, \quad y = \frac{u}{v} \quad \text{……….} (1)$$

Let Δx be the increment given to x and $\Delta u, \Delta v, \Delta y$ be the corresponding increments in u, v, y respectively.

Then, $y + \Delta y = \frac{u+\Delta u}{v+\Delta u}$ ....(2)

(2) – (1) gives,

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v}$$

$$i.e, \Delta y = \frac{v(u + \Delta u) - u(v + \Delta v)}{(v + \Delta v)v}$$

$$= \frac{uv + v\Delta u - uv - u\Delta v}{v^2 + v\Delta v}$$

$$= \frac{v\Delta u - u\Delta v}{v^2 + v\Delta v}$$

$$\frac{\Delta y}{\Delta x} = \frac{v\frac{\Delta u}{\Delta x} - u\,\frac{\Delta v}{\Delta x}}{v^2 + v\Delta v}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left\{\frac{v\frac{\Delta u}{\Delta x} - u\,\frac{\Delta v}{\Delta x}}{v^2 + v\Delta v}\right\}$$

$$\frac{\lim\limits_{\Delta x \to 0} \left(v\frac{\Delta u}{\Delta x} - u\,\frac{\Delta v}{\Delta x}\right)}{\lim\limits_{\Delta x \to 0} (v^2 + v\Delta v)}$$

$$= v\,\frac{\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Since $\Delta v \to 0\ as\ \Delta x \to 0\ as\ v(x)$is a continuous function.

$$\therefore\ \frac{dy}{dx} = \frac{v\frac{du}{dx}}{v^2}$$

$$i, e.\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\ \frac{du}{dx}}{v^2}$$

$$\frac{d}{dx}\frac{(Numerator)}{Denominator} = \frac{(D_r)\frac{d}{dx}\,(N_r) - N_r\,\frac{d}{dx}(\,D_r)}{(D_r)^2}$$

Where N_r $\equiv$ Numerator and D_r $\equiv$Denominator.

Illustration:

Differentiate: $y = \frac{2x+1}{x}$

Solution:

$\frac{dy}{dx} = \frac{\frac{vdu}{dx} - \frac{udv}{dx}}{v^2}$

$y = \frac{u}{v}$; u = 2x + 1; v = x

$\frac{dy}{dx} = \frac{d\left(\frac{u}{v}\right)}{dx} = \frac{\frac{xd\,(2)}{dx} - (2x+1)\,.\frac{d\,(1)}{dx}}{x^2}$

$= \frac{x.2-(2x+1).1}{x^2}$

$= \frac{2x-2x-1}{x^2}$

$= \frac{-1}{x^2}$

Function of a Function Rule:

If we have a function y = f(x) where x is in turn a function of another variable, u, then the derivative of y with respect to u is equal to the derivative of y with respect to x times in the derivative of x with respect to u.

If y = f(x) and

$x = g(u)$

Then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

Illustration :

Differentiate $y = (x^2 + 8x - 8)^{16}$ with respect to x.

Solution:

Let $y = (x^2 + 8x - 8)^{16}$ $\quad$ $u = x^2 + 8x - 8$

Then $y = u^{16}$ $\quad$ $\frac{du}{dx} = 2x+8$

$\frac{dy}{du} = 16u^{15}$

$= 16(x^2+8x-8)^{15}$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 + 8x - 8)^{16}$$

$$= 16(x^2+8x-8)^{15}. (2x+8)$$

Function of a Function Rule or Chain Rule of Differentiation

Y	$\boldsymbol{\frac{dy}{dx}}$
$f(x)$	$\boldsymbol{\frac{dy}{du} \cdot \frac{du}{dx}}$

Illustration :

Find $\boldsymbol{\frac{d}{dx} e^{ax}}$

Solution:

Let $u = ax$ $\quad$ $y = e^u$

$\frac{du}{dx} = a$, $\quad$ $\frac{dy}{du} = e^u = e^{ax}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\therefore \frac{dy}{dx} = ae^{ax}$$

So if y = e^{3x} ,

$$\frac{dy}{dx} = 3e^{3x}$$

y = e^{-5x} ,

$$\frac{dy}{dx} = -5e^{-5x}$$

y = e^{-x} ,

$$\frac{dy}{dx} = -e^{-x}$$

y = $1/e^{4x} = e^{-4x}$,

$$\frac{dy}{dx} = -4e^{-4x}$$

$$= -4/e^{4x}$$

Simple applications to Economics

Elasticity of the function:

The elasticity of a function y = *f*(*x*) at the point *x* is the rate of proportional change in *y* per unit proportional change in *x*.

$$\frac{Ey}{Ex} = \frac{x}{y} \cdot \frac{dy}{dx}$$

Illustration

Let *y* = 3*x* – 6. Find

i) elasticity and

ii) the elasticity if *x* = 8.

Solution:

$$\frac{Ey}{Ex} = \frac{x}{y} \cdot \frac{dy}{dx}$$

$$= \frac{x}{3x-6} \cdot \frac{d(3x-6)}{dx}$$

$$= \frac{x}{3x-6}(3)$$

$= \frac{3x}{3(x-2)}$

$= \frac{x}{x-2}$

When x = 8,

$$\frac{\boldsymbol{Ey}}{\boldsymbol{Ex}} = \frac{\mathbf{8}}{\mathbf{8-2}}$$

$$= \frac{\mathbf{8}}{\mathbf{6}}$$

$$= \frac{\mathbf{4}}{\mathbf{3}}$$

Elasticity of Demand

The elasticity of demand denoted by ή is defined by ή = $-\frac{p}{q}.\frac{dq}{dp}$

Illustration

q = 32 – 4p – p^2. Find ή when p = 3.

Solution:

$$ή = -\frac{p}{q}.\frac{dq}{dp}$$

$$= -\frac{3}{32 - 4(3)-(3)2}.\frac{d(32 - 4p - p2)}{dp}$$

$$= -\frac{3}{11}(-4 - 2p)$$

$$= -\frac{3}{11}(-4 - 2(3)$$

$$= \frac{30}{11}$$

ή = $\frac{30}{11}$ or 2.727.

Marginal Revenue:

It is defined as $\frac{dR}{dq}$

$$\frac{dR}{dq} = p(1 - \frac{1}{ή})$$

Illustration

If p = 3 and ή = $\frac{30}{11}$. Find$\frac{dR}{dq}$.

Solution:

$$\frac{dR}{dq} = p(1 - \frac{1}{ή})$$

$$= 3(1\text{-}\frac{30}{11})$$

$$= 3(\frac{19}{30})$$

$$= \frac{19}{10}$$

$$= 1.9$$

Illustration

The total cost, C, of making x units of a product is, C = $0.00003x^3 - 0.045x^2 + 8x + 25000$. Find the marginal cost at 1000 units out-put.

Solution:

C = $0.00003x^3 - 0.045x^2 + 8x + 25{,}000$ ………………….(1)

Marginal Cost at the level x is $\frac{dC}{dx}$

Differentiating both sides of with respect to *'x'*

$$\frac{dC}{dx} = 0.00009\ x^2 - 0.09\ x + 8$$

Marginal Cost at $x = 1000$ is $\frac{dC}{dx}$ at $x = 1000$.

When $x = 1000 \Rightarrow \frac{dC}{dx} = 0.00009\ (1000)^2 - 0.09(1000) + 8 = 90 - 90 + 9 = 8$

∴Marginal Cost at $x = 1000$ is 8.

*****

Practical Problems:

1. Find the differential coefficient of the following functions:

   a. x^9
   b. $x^{3/2}$
   c. $\frac{1}{x}$
   d. $\frac{1}{\sqrt{x}}$
   e. $2x$
   f. $x^{-5/2}$
   g. $x^{\log 2}$
   h. 2^3

   Answers:

   a. $9x^8$
   b. $\frac{3}{2}x^{1/2}$
   c. $-\frac{1}{x^2}$
   d. $-\frac{1}{2x\sqrt{x}}$
   e. 2
   f. $-\frac{5}{2}x^{-7/2}$
   g. $(\log 2)\, x^{(\log 2) - 1}$
   h. 0

2. Differentiate the following with respect to 'x':

   a. $\frac{2}{3}x$

   b. $-x^2$

   c. $2e^x$

   d. $x^5.e^x$

   e. $x^2 \log x$

   f. $e^x \log x$

g. $\frac{2x+1}{3x-2}$

Answers:

a. $\frac{2}{3}$

b. $-2x$

c. $2e^x$

d. $x^4 e^x (x+5)$

e. $x (2 \log x + 1)$

f. $e^x (\log x + \frac{1}{x})$

g. $-\frac{7}{(3x-2)^2}$

3. Differentiate the following with respect to x:

a. $(5-2x)^4$

b. $(3x^2 - 8x + 9)^2$

Answers:

a. $-8(5-2x)^3$

b. $4(3x^2-8x+9)(3x-4)$

4. The total cost C of making x units of product is C = 50 + 3x + $\sqrt{x}$. Find the marginal cost at 100 units output. (Ans. 3.05)

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UNIT- IV

HIGHER ORDER DERIVATIVES

Higher order derivatives – maxima and minima – simple marketing models using profit maximization, fencing and container problems only – Integral calculus – standard forms – rules of integration – Definite integral – simple applications – finding total and average cost function – producer surplus and consumer surplus.

Higher Order Derivatives:

The process of differentiation can be applied several times in succession, leading in particular to the second derivative f″ of the function f, which is just the derivative of the derivative f′. The second derivative often has a useful physical interpretation.

$\frac{d}{dx}(\frac{dy}{dx})$ is denoted by $\frac{d^2y}{dx^2}$ read as:

"*d* – squared *y* by *dx* squared". This is called the second order differential coefficient of *y* with respect to *x.*

Various symbols used to denote second order derivative $(D^2 = \frac{d^2y}{dx^2})$ are y^2, y'', *y(2)*, $y''(x)$, $f^{(2)}(x)$, D^2y.

Similarly $\frac{d^3y}{dx^3}$ is the third order differential coefficient of y with respect *x* and so on.

Concept of second and higher order derivatives is needed in optimization problems.

Example: 1

Consider the function $y = x^5$

Solution:

$$y = x^5 \qquad \text{....... (1)}$$

$$\frac{dy}{dx} = 5x^4 \qquad \text{....... (2)}$$

$$\frac{d^2y}{dx^2} = 20\,x^3 \qquad \text{....... (3)}$$

$$\frac{d^3y}{dx^3} = 60\,x^2 \qquad \text{.......}\ (4)$$

$$\frac{d^4y}{dx^4} = 120\,x \qquad \text{.......}\ (5)$$

$$\frac{d^5y}{dx^5} = 120 \qquad \text{.......}\ (6)$$

Example : 2

Find $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$ if $y = 5x^3 - 7x$

Solution:

$$y = 5x^3 - 7x$$

$$\frac{dy}{dx} = 15x^3 - 7x$$

$$\frac{d^2y}{dx^2} = 30\,x$$

$$\frac{d^3y}{dx^3} = 30$$

Example : 3

If $y = x\,e\,x^2$ find $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.

Solution:

$$y = x\,\mathrm{e}\,x^2$$

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = x.\mathrm{e}x^2 2x + 1.\mathrm{e}x^2$$

$$= 2x^2\mathrm{e}x^2 + \mathrm{e}x^2$$

$$= \mathrm{e}x^2(2x^2 + 1)$$

$\frac{d^2y}{dx^2}$ = ex^2.2x (2x^2+1) + ex^2.4x

= ex^2(4x^3+ 2x + 4x)

= ex^2(4x^3 + 6x)

$\frac{d^3y}{dx^3}$ = ex^2 2x (4x^3 + 6x) + ex^2 (12x^2 + 6)

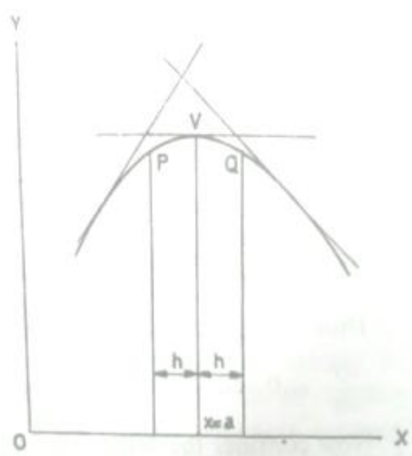
= ex^2(8x^4 + 12x^2 + 12x^2 + 6)

= ex^2(8x^4 + 24x^2 + 6)

MAXIMA AND MINIMA:

CRITERIA FOR MAXIMA

Consider the curve y = f(x) given in the below Fig. From the figure we see that V is the highest point of the curve and so the y co - ordinate of V is the maximum value that y can take. The ordinates of the points on either side of V are all less than the ordinate of V.



Definition

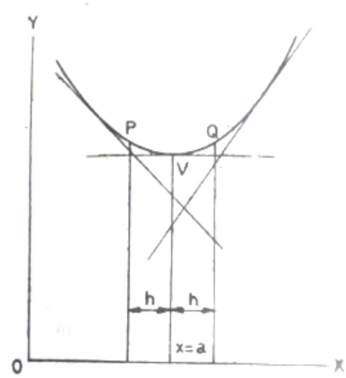
A function y = f (x) is said to have a **maximum** at x = a if f (a + h) – f (a) $\leq$ 0 for sufficiently small h positive or negative. So, we find that the function y = f (x) increases as x increases upto V, and then decreases as x increases after crossing V. It is clear from figure that the value of $\frac{dy}{dx}$will be positive up to V, becomes zero at V and will be negative after crossing V. Therefore its differential coefficient $\frac{d^2y}{dx^2}$ should be negative at V.

Thus we have, $\frac{dy}{dx} = y' = 0$ and

$\frac{d^2y}{dx^2}$ is negative at a point where y is maximum.

CRITERIA FOR MINIMA

Consider the curve y=f(x) given in the below figure. From the figure we see that V is the lowest point of the curve and so the y - coordinate of V is the minimum value that y can take. From Figure we see that the ordinates at the points on either side of V are all greater than the ordinate of V.



Definition

A function y = f (x) is said to have a **minimum** at x = a if f (a + h) – f (a) ≥ 0, for sufficiently small h, positive or negative. So, we find that the function y = f (x) decreases as x increases upto V and th en increases as x increases after crossing V.

It is clear that the value of $\frac{dy}{dx}$ will be negative upto V, becomes zero at V and will be positive after crossing V. Therefore its differential coefficient $\frac{d^2y}{dx^2}$should be positive at V.

Thus we have, $\frac{dy}{dx}$ = y' = 0 and

$\frac{d^2y}{dx^2}$ is positive at a point where y is minimum.

POINTS OF INFLEXION

Consider the below figure. Here at the point V the tangent is parallel to the x - axis and therefore $\frac{dy}{dx} = 0$. But V gives neither a maximum point nor a minimum point. Actually the tangent crosses the curve here. At such a point $\frac{d^2y}{dx^2} = 0$ and such points are called **points of inflexion**.

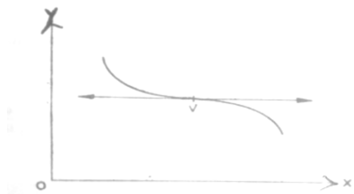


Fig.

Extreme Values:

The maximum and minimum values of a function are called the **extreme values** of the function.

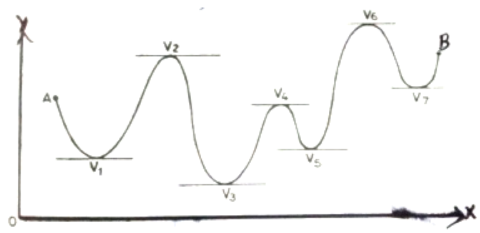


Fig. Maxima and Minima

Step by Step Procedure to find Maxima and Minima of a Function of one Variable

1. Find $\frac{dy}{dx}$ for the given function y = f(x)
2. Find the value or values of x which make $\frac{dy}{dx}$ zero. Let these be x_1, x_2, x_3, ...
3. Find $\frac{d^2y}{dx^2}$.
4. Find the sign of $\frac{d^2y}{dx^2}$ at $x = x_1, x_2,\ldots$ And hence decide which of these x_1, x_2, ... maximize or minimize the function.
5. Find the maximum and minimum values of the function by substituting for x in y = f(x) by choosing the suitable value of x from x_1, x_2, x_3, ...

Sometimes, by the very nature of the problem, it will be clear whether the value of x which makes $\frac{dy}{dx}$ zero maximizes y or minimizes y. In such cases there is no need for the second derivative test.

We tabulate the result as:

	Maximum	**Minimum**
Necessary condition	$\frac{dy}{dx} = 0$	$\frac{dy}{dx} = 0$
Sufficient condition	$\frac{dy}{dx}=0; \frac{d^2y}{dx^2} < 0,$	$\frac{dy}{dx}=0, \frac{d^2y}{dx^2} > 0$

Illustration:

Examine the cost function, $y = 40 - 4x + x^2$ for maximum or minimum.

Solution:

$y = 40 - 4x + x^2$ ……(1)

$\frac{dy}{dx} = -4 + 2x$ …….(2)

$\frac{dy}{dx} = 0$ if $-4 + 2x = 0 \Rightarrow x = 2$

Differentiating (2), with respect to x,

$$\frac{d^2y}{dx^2} = 2 > 0 \text{ always}$$

$\therefore$ $x = 2$ makes y, a minimum and this minimum value of y is got by putting $x = 2$ in (1).

$$y = 40 - 4 \times 2 + 2^2 = 40 - 8 + 4 = 36.$$

Illustration

Examine the function, $y = 2x^2 - x^3 + 5$for maximum and minimum.

Solution:

$y = 2x^2 - x^3 + 5$ ……(1)

$\frac{dy}{dx} = 4x + 3x^2$ …….(2)

$\frac{dy}{dx} = 0$ gives, $4x + 3x^2 = 0 \Rightarrow x = 0$ or $\frac{4}{3}$.

Differentiating (2), with respect to x,

$$\frac{d^2y}{dx^2} = 4 - 6x$$

At $x = 0$, $\frac{d^2y}{dx^2} = 4 – 0 = 4 > 0$

∴ $x = 0$ makes y, a minimum and this minimum value of y is got by putting $x = 0$ in (1) is 5.

At $x = \frac{4}{3}$, $\frac{d^2y}{dx^2} = 4 – 6(\frac{4}{3}) = - 4 < 0$

∴ $x = \frac{4}{3}$ makes y, a maximum and this maximum value of y is got by putting $x = \frac{4}{3}$ in (1) is $\frac{167}{27}$.

Illustration:

Examine for extrema, if any, for the function $y = x^5 – 5x^3 + 6$. (x is real).

Solution:

$y = x^5 – 5x^3 + 6$

$\frac{dy}{dx} = 5x^4 + 15 x^2$ …….(2)

$\frac{dy}{dx} = 0$ gives,

$5x^4 + 15 x^2 = 0$

$x = 0.$

Since x is real $x^2 + 3 \neq 0$

$\frac{d^2y}{dx^2} = 20x^3 + 30 x$

When $x = 0$,

$\frac{d^2y}{dx^2} = 0$

$\frac{d^2y}{dx^2} = x(20x^2 + 30)$ and is less than zero if x is less than zero.

Also $\frac{d^2y}{dx^2}$ is $0 >$ if $x > 0$

$\frac{d^2y}{dx^2}$ changes sign when it passes through $x = 0$

Hence $x = 0$ is a **point of inflexion**.

Simple Marketing Models Using Profit Maximization

Let C (q) be the cost of producing q units of an item and S (q) the sales revenue got by selling those q unit.

Then we show that when the profit is maximum marginal sales is equal to marginal cost.

Let P (q) be the profit function.

Then, $P(q) = S(q) - C(q)$

Differentiating with respect to q,

$P'(q) = S'(q) - C'(q)$

When the profit is maximum ,

$$P'(q) = 0$$

$$\therefore \; S'(q) - C'(q) = 0$$

$$\therefore S'(q) = C'(q)$$

Marginal sales = Marginal cost.

Since the costs of production will increase indefinitely as q increases and the market is limited, if there is a turning point for $P(q)$, it will yield a maximum.

Now we turn our attention to pricing problems.

Let q be the quantity of a certain item sold by a firm and

Let P be the sales price per unit.

Let $C(q)$ be the total cost of production and P the net profit function.

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

$$\therefore \qquad \text{P} = pq - \text{C}(q)$$

Suppose that the manager wishes to find the price that will maximise the net profit. Suppose the production function C(q) and the quantity q are known in terms of p, the price, the profit can be expressed as the function of a single variable p ; and the optimal price can be determined accordingly.

Example

A manufacturer can sell x items per month at a price of p = 300 – 2x rupees. Producing x items costs the manufacturer y rupees where y = 20x + 1000. How much production will yield maximum profits?

Solution:

The price of one item = p = Rs (300 – 2x)

∴ The revenue got by selling x items

$$= x(300 - 2x)$$

$$= \text{Rs. } 300x - 2x^2$$

The cost of producing x items = Rs. (20x+1000)

Let P be the profit function.

$$P = \text{Revenue} - \text{Cost}$$

$$= 300x - 2x^2 - (20x + 1000)$$

$$= -2x^2 + 280x - 1000$$

$$\frac{dP}{dx} = -4x + 28$$

$\frac{dP}{dx} = 0$ gives

$- 4x + 280 = 0$

i.e. $x = 70$

$\frac{d^2P}{dx^2} = - 4 < 0$

So P is maximum when $x = 70$.

Hence when 70 items are produced and sold the profit is maximum.

Applications: (Fencing And Container Problems Only)

Example 1

A rectangular field is y metres long, x metres wide, What is the minimum amount of fence which will enclose 10,000 square metres.

Solution:

Let A denote the area and

F the perimeter of the field.

We know that,

$A = xy$ and

$F = 2x+2y$

Now it is given,

$xy = 10,000$ …………. (1)

Our problem is to minimize F subject to the constraint (1).

We find that F depends upon two variables x and y;

But in fact, F is essentially a function of one variable only since x and y are connected by the relations (1).

Writing y in terms of x,

$$y = \frac{10,000}{x}$$

Substituting this value of *y* in the formula for *F*.

$F = 2x + \frac{20,000}{x}$ …………………….. (2)

The value or values of *x* which optimizes *F* are given by solving the equation,

$$\frac{dF}{dx} = 0.$$

Differentiating (2) with respect to *x*,

$$\frac{dF}{dx} = 2 - \frac{20,000}{x^2}$$

$$\frac{dF}{dx} = 0 \; gives,$$

$$2 - \frac{20,000}{x^2} = 0$$

$$2x^2 = 20,000$$

$$x^2 = 10,000$$

$$x = \pm 100$$

x = - 100 is meaningless and

x = + 100 is the only admissible value of *x*.

Differentiating (3) with respect *x*,

$$\frac{d^2F}{dx^2} = -20,000 \times \left(-\frac{2}{x^3}\right)$$

$$= \frac{40,000}{x^3}$$

When $x = 100$,

$$\frac{d^2F}{dx^2} = \frac{40,000}{100^3} > 0$$

$x = 100$ minimises F

Now $y = \frac{10,000}{x}$

When $x = 100$,

$$y = \frac{10,000}{100} = 100$$

Thus $x = 100$, $y = 100$ are the dimensions which make F minimum.

$$F = 2x+2y$$

$$F_{MIN} = 2 \text{ x } 100 + 2 \text{ x } 100 = 400 \text{ metres.}$$

Note.

In fact the rectangle enclosing a prescribed area will have the least perimeter if it is a square. Similarly, given the perimeter of a rectangle it will enclose the largest possible area if it is a square.

Example 2.

A cylindrical can with no top is to have a volume of 25 cubic centimetres. If the surface area is to be a minimum, what should be the height and base radius of the can?

Solution:

Let V be the volume and

S the surface area of the can (without lid.)

If r is the base radius and

h, the height,

$V = \pi r^2 h$ `and

$S = \pi r^2 + 2\pi rh$ ……………..(1)

But V = 25 (given)

$\pi r^2 h = 25$ ………………. (2)

Our problem is to minimize S in (1) subject to the constraint (2),

Substituting the value for h $=\frac{25}{\pi r^2}$ got from (2) in (1),

S can be expressed as a function of a single variable *r*.

$$S = 2\pi r \left(\frac{25}{\pi r^2}\right) + \pi r^2$$

$$= \frac{50}{r} + \pi r^2 \quad \text{………………. (3)}$$

$$\frac{ds}{dr} = \frac{50}{r^2} + 2\pi r$$

$$\frac{ds}{dr} = 0 \; gives,$$

$$-\frac{50}{r^2} + 2\pi r = 0$$

$$2\pi r^3 = 50$$

$$r^3 = \frac{25}{\pi}$$

$$r = \left(\frac{25}{\pi}\right)^{1/3}$$

$$\frac{d^2S}{dr^2} = \frac{2 \text{ x } 50}{r^3} + 2\pi$$

$$= \frac{100}{r^3} + 2\pi$$

When $r = \left(\frac{25}{\pi}\right)^{1/3}$ $\frac{d^2S}{dr^2}$ is positive

and therefore $r = \left(\frac{25}{\pi}\right)^{1/3}$ makes S minimum

Now h = $\frac{25}{\pi r^2}$ from (2)

When $r = \left(\frac{25}{\pi}\right)^{\frac{1}{3}}$

$$h = \frac{25}{\pi\left(\frac{25}{\pi}\right)^{2/3}}$$

$$= \left(\frac{25}{\pi}\right)^{1/3}$$

Substituting these values of r and h in (1), we get the minimum value for S.

Note.

(i) Is the second derivative test necessary to prove that r = $\left(\frac{25}{\pi}\right)^{1/3}$ minimises S?

Is it not obvious from practical considerations?

(ii) If the box has a lid,

$$S = 2\pi rh + \pi r^2$$

INTEGRAL CALCULUS

Integral:

The technique of finding the functional relationship is called "Integration". It is the reverse or inverse of differentiation.

Integral calculus:

Integral calculus deals with finding a function when its rate of change is known.

Constant:

A constant is a quantity which has only one value throughout a particular investigation. The constant can be classified into two classes. They are:

i) Absolute constant and

ii) Arbitrary constants

Absolute constant:

It is constant whose value never changes.

Arbitrary constants:

It is a constant whose value may change from problem to problem but remains fixed in a single problem.

For example : a, b, c, …. etc.

Definition:

A function *F(x)* is called as anti derivative or integral of a function *f(x)* on an interval I if,

F'(x) = f(x) for every value of *x* in I

If the differential coefficient of F(x) with respect x is f(x), then the integral of f(x) with respect to x is F(x), in symbols.

$$\text{If } \frac{dF(x)}{dx} = f(x)$$

$$\text{Then, } \int f(x)dx = F(x)$$

$\int$ is used to denote summation. The differential dx tells us that we have to integrate with respect to x. The variable x in dx is called variable of integration or integrator. The function f(x) is called integrand. The process of finding the integral is called integration.

Constant of Integration

Consider the following examples.

$$\text{i.) } \int 5\, dx = 5x + c$$

$$\text{ii.) } \int 2x\, dx = x^2 + c$$

c in the present case is called the constant of integration. Since c can take any constant value, the integral is called indefinite integral. If c assumes particular values then the integral becomes definite integral.

Standard Forms

a.) $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \; (n \neq -1)$

b.) $\int \frac{1}{x} dx = \log x + c$

c.) $\int e^x \, dx = e^x + c$

d.) $\int e^{-x} \, dx = -e^x + c$

e.) $\int e^{ax} \, dx = \frac{e^{ax}}{a} + c$

f.) $\int dx = x + c$

g.) $\int a^x \, dx = \frac{a^x}{log \; a} + c$

Illustration:

Evaluate the following

a.) $\int x \, dx = \int \frac{x^{1+1}}{1+1} + c$

$= \int \frac{x^2}{2} + c$

b.) $\int x^2 dx = \int \frac{x^{2+1}}{2+1} + c$

$= \frac{x^3}{3} + c$

c.) $\int \frac{1}{x^3} \, dx = \int x^{-3} dx$

$= \int \frac{x^{-3+1}}{-3+1} + c$

$= \int \frac{x^{-2}}{-2} + c$

$= \frac{-1}{2x^2} + c$

d.) $\int \sqrt{xdx} = \int x^{1/2} \, dx$

$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$

$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= \frac{2x^{3/2}}{3} + c$

e.) $\int x^{-4}\,dx = \frac{x^{-4+1}}{-4+1} + c$

$= \frac{x^{-3}}{-3} + c$

$= \frac{-1}{3x^3} + c$

f.) $\int 4x^3\,dx = 4\int x^3\,dx$

$= 4\left[\frac{x^{3+1}}{3+1}\right] + c$

$= \frac{4x^4}{4} + c$

$= x^4 + c$

g.) $\int \frac{1}{x+5}\,dx = \log(x+5) + c$

h.) $\int e^{2x}\,dx = \frac{e^{2x}}{2} + c$

Illustration :

Let x represent the investment in rupees and $\mathrm{g}(x) = \frac{x}{20}$ be the marginal out-put of paddy. Find the total out-put function denoted by $f(x)$ when no paddy can be produced with-out any investment. What will be the out-put function if the land is in such a bad condition that only after spending Rs. 1000 for clearing it, one can start farming?

Solution:

$$\mathrm{g}(x) = \frac{x}{20}$$

$$f(x) = \int \frac{x}{20}\,dx = \frac{x^2}{40} + c$$

$$f(x) = \frac{x^2}{40} + c \qquad \text{.... (1)}$$

(i) Given that when $x = 0, f(x) = 0$. Using this in (1) we have

$$0 = 0 + c \qquad i.e \quad c = 0$$

Hence $\qquad f(x) = \frac{x^2}{40}$

(ii) Given that when $x = 1000, f(x) = 0$

Using this in (1) we have

$$0 = \frac{1000^2}{40} + c$$

$$c = -\frac{1000^2}{40} = -25000$$

$$f(x) = \frac{x^2}{40} - 25,000$$

Integration is inherently more difficult than differentiation. As we have done in differentiation we will have lists of standard forms and certain rules of integration using which we shall be in a position to integrate functions which normally arise in business situations.

Standard Forms List I

(1) $\int x^n \ dx = \frac{x^{n+1}}{n+1} + c$

Where n is any constant other than -1

For, $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + c\right) = \frac{1}{n+1} \ \frac{d}{dx} x^{n+1}$

$$= \frac{1}{n+1}(n+1)x^{n+1-1} = x^n$$

(2) If $\quad n = -1,$

$$\int x^n dx = \int x^{-1} dx = \int \frac{1}{x} dx$$

$$= \log \ x + c$$

since $\frac{d}{dx} \ (\log \ x + c) = \frac{1}{x}$

(3) $\quad \int e^{kx} \ dx \ = \frac{e^{kx}}{k} + c$ where k is any constant

For, $\frac{d}{dx}\left(\frac{e^{kx}}{k} + c\right) = \frac{1}{k} \ \frac{d}{dx} \ e^{kx}$

$$= \frac{1}{k} \times ke^{kx} \ = e^x$$

$\therefore \quad \int e^{kx} \ dx = \frac{e^{kx}}{k} + c$

(4) $\int a^{kx}\, dx = \frac{a^{kx}}{k \log\, a} + c$

Where, ‘k’ and ‘a’ are constants

For, $\frac{d}{dx}\left(\frac{a^{kx}}{k \log\, a}\right) = \frac{1}{k \log\, a} \frac{d}{dx} a^{kx}$

$= \frac{1}{k \log\, a}\; a^{kx}\;\; k \log a = a^{kx}$

$\therefore \quad \int a^{kx}\, dx = \frac{e^{kx}}{k\ log\ a} + c$

Particular useful cases of these standard forms:

$$\int dx = \int 1\, dx = \int x^o\, dx$$

$= \frac{x^{o+1}}{o+1} + c = x + c$

This is a particular case of the standard form (1) when, $n = 0$. Even direct method of evaluation is very simple once we observe that

(a) $\int dx = \int 1.\ dx\ and\ \frac{d}{dx}(x) = 1$

(b) $\int e^x\, dx = e^x + c$

particular case of the standard form (3) when k = 1

(c)$\int a^x\, dx = \frac{a^x}{log\ a} + c$ is also a particular case of (4)

when k = 1

Some Rules of Integration

(1) $\int \{f_1(x) + f_2(x)\} dx = \int f_1(x) dx + \int f_2\,(x) dx$

This can be extended to a sum of any finite number of functions. This rule states that integral of a sum of any finite number of functions is equal to the sum of the integrals of the individual functions.

Similarly we have

$\int\{f_1(x) - f_2(x)\}\ dx$

$$= \int f_1(x)\,dx - \int f_2\ (x)\,dx$$

and $\int f_1\ (x) = f_2(x) + f_3(x)\}\,dx$

$$= \int f_1(x)\,dx - \int f_2\ (x)\,dx + \int f_3\ \ (x)\,dx$$

and so on.

(2) $\int k\ f(x)dx = k \int f(x)dx$ Where k is any constant.

(Proofs can be supplied by using the corresponding rules we have in differentiation)

Illustration :

Evaluate $\int x^{1/2}\ dx$

Solution:

We know that $\int x^n\ \ dx\ \ = \frac{x^{n+1}}{n+1} + x\ \ if\ n \neq -1$

$$\int x^{1/2}\ dx = \frac{x^{1/2+1}}{\frac{1}{2}+1} + c$$

$$= \frac{x^{3/2}}{\frac{3}{2}} + c = \frac{2}{3} - x^{3/2} + c$$

Illustration :

Evaluate $\int(x^3 + 4x^2 - 5x - 6)\ dx$

Solution:

$$\int (x^3 + 4x^2 - 5x - 6)\ dx = \int x^3\ dx + \int 4x^2\ dx - \int 5x\ dx - \int 6dx$$

$$= \frac{x^{3+1}}{3+1} + 4\int x^2\ dx - 5\int x\ dx - 6\int dx + c$$

$$= \frac{x^4}{4} + 4\frac{x^{2+1}}{2+1} - 5\ \frac{x^{1+1}}{1+1} - 6x + c$$

$$= \frac{x^4}{4} + \frac{4}{3}\ x^3 - 5\frac{x^2}{2} - 6x + c$$

Illustration :

Evaluate $\int\left(x^{3/2}-\frac{1}{x^2}+\frac{5}{x}-7\right)\,dx$

Solution:

$$\int\left(x^{3/2}-\frac{1}{x^2}+\frac{5}{x}-7\right)\,dx=\int x^{3/2}\,dx-\int x^{-2}\,dx+\int\frac{5}{x}\,dx-\int 7dx$$

$$=\frac{x^{3/2+1}}{\frac{3}{2}+1}-\int x^{2}\,dx+5\int\frac{1}{x}\,dx-7\int dx+c$$

$$=\frac{x^{5/2}}{\frac{5}{2}}-\frac{x^{-2+1}}{-2+1}+5\,log\,x-7x+c$$

$$=\frac{2}{5}\,x^{5/2}+x^{-1}+5\,log\,x-7x+c$$

Illustration :

Evaluate $\int\left(3^{x}+e^{x}+x^{-7/2}\right)dx$

Solution:

$$\int\left(3^{x}+e^{x}+x^{-7/2}\right)\,dx=\int 3^{x}\,dx+\int e^{x}\,dx+\int x^{-7/2}\,dx$$

$$=\frac{3^{x}}{log\;3}+e^{x}+\frac{x^{-7/2+1}}{-\frac{7}{2}+1}+c=\frac{x}{log\;3}+e^{x}-\frac{2}{5}\,x^{-5/2}+c$$

Illustration :

Evaluate $\int\left(\frac{4x^{7}+3x^{3}-5x^{2}}{x^{4}}\right)\,dx$

Solution:

$$\int\left(\frac{4x^{7}+3x^{3}-5x^{2}}{x^{4}}\right)\,dx=\int\frac{4x^{7}}{x^{4}}\,dx+\int\frac{3x^{3}\,dx}{x^{4}}-\int\frac{5x^{2}\,dx}{x^{4}}$$

$$=\int 4x^{3}\;dx+\int\frac{3}{x}\,dx-\int\frac{5}{x^{2}}\,dx$$

$$=4\frac{x^{4}}{4}+3\,log\,x-5\int x^{-2}\,dx+c$$

$$= x^4 + 3\ log\ x - 5\ \frac{x^{-2+1}}{-2+1} + c$$

$$= x^4 + 3\ log\ x + 5x^{-1} + x$$

Illustration :

Evaluate $\int \left(x + \frac{1}{x}\right)^2\ dx$

Solution:

$$\int \left(x + \frac{1}{x}\right)^2\ dx = \int \left(x^2 + 2 + \frac{1}{x^2}\right)\ dx$$

$$= \int x^2\ dx + \int 2\ dx + \int x^{-2}\ dx$$

$$= \frac{x^{2+1}}{2+1} + 2x + \frac{x^{-2+1}}{-2+1} + c \quad = \frac{x^3}{3} + 2x - x^{-1} + c$$

Illustration :

Evaluate $\int (e^{2x} - 5x^{-1})\ dx$

Solution:

$$\int (e^{2x} - 5x^{-1})\ dx = \int e^{2x}\ dx - \int 5\ x^{-1}\ dx$$

$$= \frac{e^{2x}}{2} - 5\ log\ x + c$$

Definite Integral

Consider the integral $\int 3x^2\ dx$

We know that $\int 3x^2\ dx = x^3 + c$

When $x = 4$, the value of the integral is $4^3 + x = 64 + c$

When $x = 2$, the value of the integral is $2^3 + x = 8 + c$

(The value of the integral $\int 3x^2\ dx$ when $x = 4$) - (The value of the integral $\int 3x^2\ dx$ when $x = 2$)

$$= 64 + c - (8 + c) = 64 - 8 = 56$$

We say that we have integrated the function $3x^2$ with respect to x from $x = 2$ to $x = 4$ and we denote the whole thing as

$$\int_2^4 3x^2\, dx = 56$$

$$\int_2^4 3x^2\, dx = \text{value of} \int 3x^2\, dx\ at\ x = 4$$

$$= \text{value of} \int 3x^2\, dx\ at\ x = 2$$

2 and 4 are called limits of integration. 2 is called the lower limit of the integral and 4 is called the upper limit of the integral and $\int_2^4 3x^2\ dx$ is called the **"Definite Integral"** of $3x^2$ integrated from $x = 2$ to $x = 4$. Notice that in the value of the definite integral, C, the arbitrary constant of integration disappears. That is, the integral has no arbitrariness. It has got a definite value and hence the name definite integral.

Definition

If $\int f(x)\, dx = g(x) + c$, then $\int_a^b f(x)\, dx = g(b) - g(a)$ is called the **Definite Integral of** $f(x)$ with respect to x integrated from $x = a$ to $x = b$, 'a' and 'b' are called the lower and upper limits of integration respectively.

We denote $\mathrm{g(b) - g(a)}$by $[g(x)]_a^b$

$\therefore \quad \int_a^b f(x)\, dx = [g(x)]_a^b$

It is easy to verify the following facts

$(i) \quad \int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

$(ii) \quad \int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx, a < c < b$

$$= \int_{a}^{c_1} f(x)\, dx + \int_{c_1}^{c_2} f(x)\, dx + \cdots + \int_{c_{n-1}}^{b} f(x)\, dx$$

Where $a < c_1 < c_2 \ldots \ldots < c_{n-1} < b$

Proofs follow at once from the definition of the definite integral.

Meaning of a Definite Integral

Consider $\int_{2}^{4} 4x \ dx$

$$\int_{2}^{4} 4x \ dx = [2x^2]_2^4 = 32 - 8 = 24$$

Has the number 24 any significance? Consider the graph of the integrand $i.e., 4x$. Let $y = 4x$. We know that the graph of $y = 4x$ is a straight line. Consider the portion of the graph lying between $x = 2$ (the lower limit of the integral) and $x = 4$ (the upper limit of the integral).

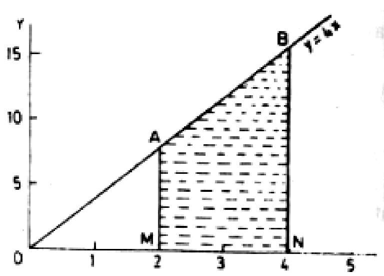


Fig. 1.

Consider the area included by the part of this line bounded by the ordinates at $x = 2$ (the lower limit of the definite integral) and at $x = 4$ (the upper limit of the definite integral) and the part of the x - axis lying between these ordinates as shown in Fig. 1.

The area bounded by the stated lines is the trapezium AMNBA. We know that the area of AMNB

$$= \frac{1}{2}(AM + BN)MN$$

$$= \frac{1}{2}(8 + 16) \times 2 = 24$$

Thus we notice that $\int_2^4 4x \ dx = 24$, is the area bounded by $y = 4x$, the ordinates at $x = 2, \ x = 4$, and the part of the $x -$ axis lying between them. This is not a coincidence. This is in fact generally true. The definite integral $\int_a^b f(x) \, dx$ is the area bounded by the curve $y = f(x)$, the ordinates at $x = a, x = b$ and the x - axis cut off by these ordinates.

Area under a Curve

Let us find the area below the curve $y = f(x)$, bounded by the $x -$ axis and the lines $x = a, x = b$.

Divide the length $MN(= b - a)$ into n equal parts each of length Δx.

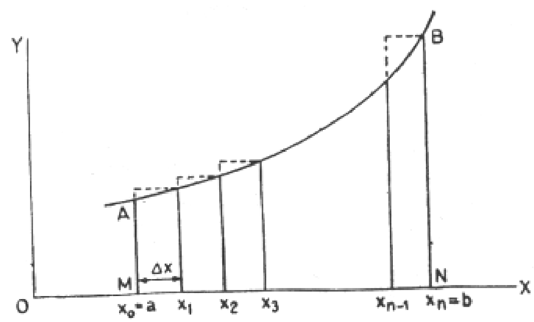


Fig. 2

Then $\Delta^x = \frac{b-a}{n}$

The points of subdivision are

$$x_0 = a$$

$$x_1 = a + \Delta x$$

$$x_2 = a + 2\Delta x$$

$$x_3 = a + 3\Delta x$$

$$x_n = a + n\Delta x$$

Let the area of AMNB be A.

The area AMNB to be found out can be approximated by the sum of the areas of n rectangles as follows.

$$A = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n).\Delta x \text{ nearly}$$

$i.e.,$ $$A = \sum_{i=1}^{n} f(x_i)\,\Delta x \text{ nearly}$$

We will have better approximation if we have finer subdivisions i.e., more rectangles with still smaller width and thus,

$$A = \underset{n \to \infty}{Lt} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

Area by Integration: The Definite Integral

The limit of the sum of areas can be evaluated by integration. Let us find the area bounded by the curve$y = f(x)$, the lines $x = a, x = b$ and the $x -$ axis as shown in the figure.

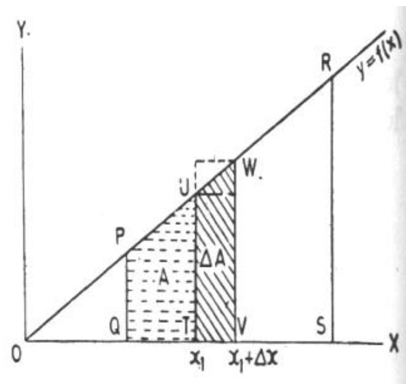


Fig. 3

(We take the curve to be a straight line for simplicity sake) We can consider the area $PQRS$ as that traced out by moving PQ parallel to itself in such a way that P moves along the curve $y = f(x)$ and Q moves along the $x -$ axis starting from the position PQ marked in the figure to the position RS. Let TU be the position of PQ at some general stage. Let x_1 be the $x -$ coordinate of T and therefore $TU = f(x_1)$. Let the area $PQTU$ be A and let VW be another position of PQ such that it is very close to TU. Let $TV = \Delta x$ and the area $UTVW$ beΔA. It is clear that the area is a function of x_1 for if we vary x_1 the area A will vary in general and ΔA is nothing but the increase in area A as x_1 increases by Δx to$x_1 + \Delta x$. Apparently ΔA is larger than the rectangle which is determined by the base Δx times the height $f(x_1)$ and smaller than the rectangle determined by Δx times$f(x_1 + \Delta x)$.

i.e., $$f(x_1)\Delta x < \Delta A < f(x_1 + \Delta x)\Delta x$$

Dividing this inequality throughout by Δx and noting that Δx is positive we have

$$f(x_1) < \frac{\Delta A}{\Delta x} < f(x_1 + \Delta x) \qquad \text{....(1)}$$

Since we consider continuous function only

$$f(x_1 + \Delta x) \to f(x_1) \qquad \text{as } \Delta x \to 0$$

and also $$\underset{\Delta x \to 0}{Lt} \frac{\Delta A}{\Delta x} = \frac{dA}{dx}$$

Taking limit as $\Delta x \to 0$ of (1)

$$f(x_1) < \underset{\Delta x \to 0}{Lt} \frac{\Delta A}{\Delta x} < f(x_1)$$

$$\underset{\Delta x \to 0}{Lt} \frac{\Delta A}{\Delta x} = f(x_1)$$

i.e., $$\frac{dA}{dx} = f(x_1)$$

Since x_1 is a general value of x which can be anywhere between a and b we can write

$$\frac{dA}{dx} = f(x)$$

$$dA = f(x)dx$$

$$A = \int f(x)\,dx + c$$

Let $$\int f(x)\,dx = g(x)$$

Then $$A = g(x) + c \qquad \text{....(2)}$$

From the diagram we see that A represents the area under the curve $y = f(x)$ and between the lines $x = a$ and $x - x_1$.

When $x_1 = a$ (its initial value), A = 0 trivially.

(2) becomes $0 = g(a) + c$

$c = -g(a)$ ....(3)

Using (3) in (2) we have

$$A = g(x) - g(a) = \int_a^x f(x)\, dx \quad \text{....(4)}$$

When $x = b$ we get the whole area $PQRS$. From (2) we have

$A = g(b) + c = g(b) - g(a)$ using (3).

$$\therefore \text{ Reqd. area } PQRS = A = \int_a^b f(x)\, dx.$$

Combining the results in this and the previous articles we have the following important result which illustrates the fact that integration is 'after all' an 'infinite summation'

$$A = \underset{n \to \infty}{Lt} \sum_{i=1}^{n} f(x_4)\, \Delta x = \int_a^b f(x)dx$$

Illustration :

$$\text{Evaluate} \quad \int_0^2 e^{2x}\, dx$$

Solution:

$$\int_0^2 e^{2x}\, dx = \left(\frac{e^{2x}}{2}\right)_0^2 = \frac{e^4}{2} - \frac{e^0}{2}$$

$$= \frac{e^4}{2} - \frac{1}{2} = \frac{1}{2}(e^4 - 1)$$

Illustration 11: Evaluate $\int_1^5 \left(x - \frac{2}{x}\right) dx$

Solution:

$$\int_{1}^{5}\left(x-\frac{2}{x}\right)dx=\left(\frac{x^2}{2}-2\log_e x\right)_1^5$$

$$=\left(\frac{25}{2}-2\,log_e\,5\right)-\left(\frac{1}{2}-2\,log\,1\right)$$

$$=\frac{25}{2}-2\,log_e\,5-\frac{1}{2}$$

$$=\frac{24}{2}-2log_e 5$$

$$=12-2\,log_e\,5$$

Applications

Application of integration are many and varied. In this section we discuss about a very few simple applications of integration.

Example :

If the marginal revenue function is

R' (x) = 15 – 9x - 3x^2

Find the revenue function (x being the number of items sold)

Solution:

Let R (x) be the revenue function

R (x) = $\int R^1(x)dx = \int(15-9x-3x^2)dx$

R (x) = 15x - $\frac{9x^2}{2}-\ x^3+c$ .....(1)

When no product is sold, the revenue is zero

i.e., When x = 0, R = 0

Using this in (1), we have c = 0. Thus (1) becomes

$R(x) = + 15x - \frac{9}{2}\ x^2 - x^3$

Example :

The marginal cost (y) function for production (x). (x) is $y' = 10 + 24x - 3x^2$; and the total cost of producing one unit is Rs. 25. Find the total cost function and the average cost function.

Solution:

Let $y(x)$ be the total cost function.

Then, $y(x) = \int y'(x)dx = \int (10 + 24x - 3x^2 +)dx$

$y(x) = 10x+12x^2 - x^3 + c$

When $x = 1, y = 25$

$25 = 10+12-1+c.$

$4 = c$

Using (2) in (1) we have

$y(x) = 10x + 12x^2 - x^3 + 4$

Let c be the average cost so that

$c = \frac{y(x)}{x}$

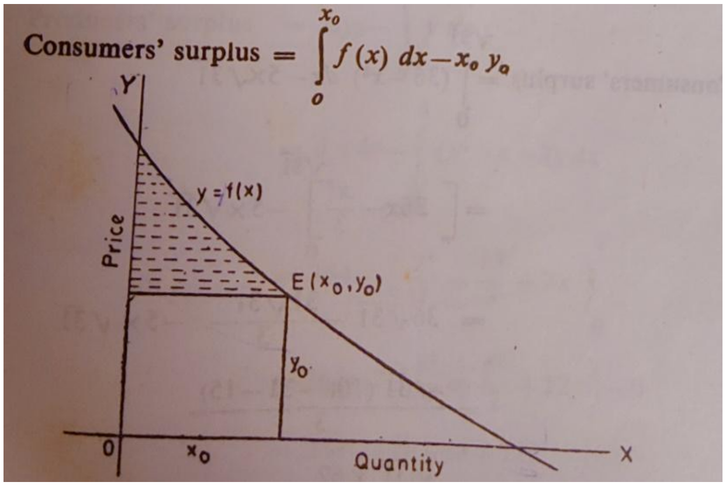
$= \frac{10x+12x^2-x^3+4}{x}$

$= 10+12x-x^2+\frac{4}{x}$

Consumer's surplus :

A demand function represents the relation between the quantity and price, i.e., the amount of commodity that can be bought at a given price. Let x_0 be the quantity bought for the price y_0. Those consumers, who are willing to pay more than the market price gain from the fact that the price is only y_0. This gain is called the consumers surplus. It is represented by the area below the demand curve and above the line $y = y_0$

Consumers surplus = $\int_0^{x_0} f(x)dx - x_0 y_o$



Example :

The demand function of a commodity is y = 36- x^2. Find the consumers surplus for (i) y_0 = 11 (ii) y_0 = 5

Solution:

Let y denote price and x the quantity demanded

Consumers surplus = $\int_0^{x_0} y\, dx - y_0\, x_0$

When the price is y_0, demand is x_0

∴ $y_0 = 36 - x_0{}^2$, ...(1)

i) When $y_0 = 11$, (1) becomes $11 = 36 - x_0{}^2$

$x_0{}^2 = 36 – 11 = 25$

$x_0 = 5$ (negative value being inadmissible)

Consumer's surplus = $\int_0^5 (36 - x^2)dx - 11 \times 5$

$= \left[36x - \frac{x^2}{3}\right]$- 55 = 36× 5 $-$ $\frac{125}{3}$ – 55

$\frac{540 - 125 - 165}{3} = \frac{540 - 290}{3} = \frac{250}{3}$

ii) When $y_0 = 5$, $5 = 36 - x_0^{\,2}$ from (1)

$x_0^{\,2} = 31$, $x_0 = \sqrt{31}$ (negative value being inadmissible)

Consumer's surplus = $\int_0^{\sqrt{31}} (36 - x^2)\,dx - 5 \times \sqrt{31}$

$= \left[36x - \frac{x^3}{3}\right] - 5 \times \sqrt{31}$

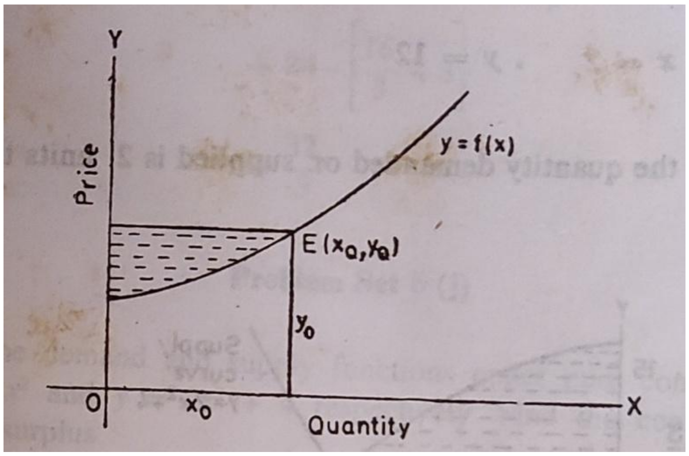
$= 36\sqrt{31} - \frac{31\sqrt{31}}{3} - 5 \times \sqrt{31}$

$= \frac{\sqrt{31}(108-31-15)}{3}$

$= \frac{\sqrt{31} \times 62}{3}$

Producers surplus :

A supply function represents the amount of a commodity that can be supplied at a price. Let y_0 be the market price and x_0 the corresponding supply. Those producers who are willing to supply the commodity below the market price gain from the fact that the price is y_0. This gain is called the producers surplus. It is represented by the area above the supply curve and below the line $y = y_0$



Example :

Find the producers surplus for the supply function $y = x^2 + x + 2$ (Where y is the price, x the quantity supplied) when $x_0 = 6$.

Solution:

$y_{0= 6^2+ 6+2=44}$

Producers surplus = $x_0 y_0 - \int_0^{x0} y\, dx$

$= 6\times 44 - \int_0^6 (x^2 + x + 2)dx$

$= 264 - \left(\frac{x^3}{3} + \frac{x^2}{2} + 2x\right)$

$= 264 - \left(\frac{6^3}{3} + \frac{6^2}{2} + 12\right) - 0$

= 264 – (72+18+12)

= 162

Example :

The demand and supply functions under pure competition are y = 16 - x^2 and y = 2x^2 + 4. Find the consumers surplus and producers surplus.

Solution:

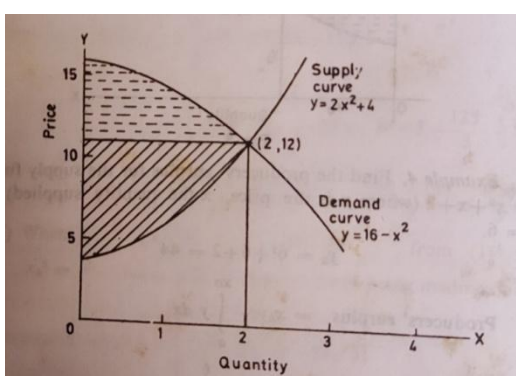
$y = 16 - x^2$ .....(1)

$y = 2x^2 + 4$ ......(2)

(2)– (1) gives $0 = 12 - 3x^2$ $x = 2$

When $x = 2$, $y = 12$

When the quantity demanded or supplied is 2 units the price is 12.



Consumers surplus = $\int_0^2 (16 - x^2)dx - 2 \times 12$

$= \left[16x - \frac{x^3}{3}\right] - 24$

$= 32 - \frac{8}{3} - 24 = \frac{16}{3}$

Producers surplus = $2\times 12 - \int_0^2 (2x^2 + 4)dx$

$= 24 - \left[\frac{2x^3}{3} + 4x\right]$

$= 24 - \left[\frac{16}{3} + 8\right]$

$= \frac{32}{3}$

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Practical Problems:

(1) Find $\frac{d^2y}{dx^2}$ for the following functions :

i) $y = \log x$ ii) $y = x^4 + 3x^2 + 6x - 2$

iii) $y = 3x - 7$ iv) $y = 5x^3 - 6x$

v) $y = 5\sqrt{x}$ vi) $y = \frac{3}{\sqrt{x}}$

vii) $y = x^2 - 5$ viii) $y = xe^x$

ix) $y = e^x \log x$ x) $y = x^2 \log x$

(2) Find $y_{1,}\ y_{2,}\ y_3, y_4$ for the following functions :

i) $y = x^2$ ii) $y = x^3$

iii) $y = 5x^4 - 7x^2$ iv) $y = e^x$

v) $y = \log x$ vi) $y = e^x \log x$

vii) $y = \frac{10}{x} + 5x - 7$ viii) $y = x^5 \log x$

ix) $y = e - 2x$ x) $y = x^2$ e - x^2

(3) Find $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$ if y = $5x^2 - 7x$. (Ans. 30)

5. Find the minimum of the function $3x^4 - 8x^3 - 90x^2$. (Ans. -1375 at x = -5, -351 at x = 3)
6. Find the maximum of the function $3x^4 - 10x^3 + 3x^2 + 12x + K$. (Ans. 8 + K)
7. Find the maximum or minimum for the function y = $x^2 - 6x$. [Ans. (3, -9), minimum]
8. Find the maximum and minimum of $2x^3 - 3x^2 - 36x + 10$.
   (x = 3 min y = -71 & x = -2 max y = -54)
9. Find the point of inflexion, of the function y = $3x^3 - 6x^2$
   (No point of inflexion)
10. Find the point of inflexion, of the function y = $x^4 - 6x^3$ (Ans. 0, 0)
11. Find the point of inflexion, of the function y = ce$-\frac{x^3}{2}$ $(1, \frac{c}{\sqrt{e}}), (-1, \frac{c}{\sqrt{e}})$
12. Evaluate the following integrals

(i) $\int \left(x+\frac{1}{x}\right)\, dx$ (ii) $\int \left(x^{3/2}+2x^2-4\right)\, dx$

(In all problems x denotes the number of items sold or produced)

13. If the marginal cost function is $c'(x)$ = $\frac{100}{x}$, find c (x) if c (16) = 100. Find the average cost function $A(x)$. Evaluate c (x), $c^1(x)$ and A (x) When x = 100.

14. The marginal cost function of manufacturing shoes is $C'(x)$ = 6 + 10x - 6x^2

15. The total cost of producing a pair of shoes is Rs. 12. Find the total and average cost function.

16. The marginal revenue $R^1(x)$ = 25 – 8x+6x^2+4x^3. Find the revenue function.

17. If the rule for the marginal function of a firm is $R^1(x)$ = 10-6x. Find the total revenue function.

18. If M = $\frac{a}{\sqrt{at+b}}$ is the marginal cost function where t is the output and if the cost of zero output is zero, find the total cost $\pi = \int_0^x M(t)\, dt$, x being the total output.

19. The demand and supply functions under pure competition are y = 14 - x^2 and y = 2x^2+4 respectively. Find the consumers and producers surplus.

20. If y = 32 - 2x^2 and y = $\frac{1}{3}x^2 + 2x + 5$ are respectively the demand and supply functions under pure competition find the sonsumers and the producers surplus.

21. The demand function of a commodity is p = 100 – 5D - D^2 (p represents price and D the quantity). Find the consumers surplus when (a) p = 16, p = 34 (c) p = 50

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UNIT -V

MATRIX

Matrices – definition – types – addition, subtraction, multiplication of matrices – inverse matrix – solving a system of simultaneous linear equations using matrix inversion technique – rank of a matrix.

MATRICES

Definition

A matrix is a rectangular array consists of elements arranged in rows and columns enclosed by a pair of brackets.

Order of Matrix

A rectangular arrangement of elements in ‘m’ rows and n columns is called a matrix of order m x n.

General format of a matrix

The general format of a matrix having ‘m’ rows and ‘n’ columns is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \ldots.. & a_{1n} \\ a_{21} & a_{22} & a_{23} \ldots\ldots & a_{2n} \\ & & . & \\ & & . & \\ a_{m1} & a_{m2} & a_{m3} \ldots\ldots & a_{mn} \end{bmatrix}_{m \times n}$$

Notations:

Every matrix is denoting by capital letters and its elements by corresponding small letters.

Types of Matrices:

The following are the types of matrix

1. **Row matrix:**

   A matrix that consists of only one row is called a row matrix.

   For example: B = [1 2 3 4]
2. **Column matrix:**

A matrix consists of only one column is called a column matrix.

For example: $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

3. **Square matrix:**

A matrix, in which the number of rows is equal to the number of columns, is called square matrix.

For example: X = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

4. **Zero matrix or Null matrix:**

A matrix in which all the elements are zero is called a null matrix.

For example: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

5. **Diagonal Matrix:**

A square matrix in which all the elements are zero, except the leading diagonal elements.

For example: A = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

6. **Scalar Matrix:**

A diagonal matrix in which all the diagonal elements are equal is called a scalar matrix.

For example: A = $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

7. **Identity / Unit Matrix:**

A square matrix is said to be unit matrix if all the elements are zero expect diagonal elements and diagonal elements are equal to 1. It is denoted by 1.

For example: A = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. Triangular Matrix:

A square matrix in which all the elements below the leading diagonal elements are zero, it is called Upper Triangular Matrix.

For example: A = $\begin{bmatrix} 13 & 5 \\ 0 & 5 & 4 \\ 0 & 0 & 9 \end{bmatrix}$

9. Lower Triangular Matrix:

A square matrix in which all the elements above the leading diagonal elements are zero is called a lower triangular matrix.

For example: A = $\begin{bmatrix} -1 & 0 & 0 \\ 8 & 1 & 4 \\ 3 & -4 & 2 \end{bmatrix}$

10. Singular and Nonsingular Matrices:

A square matrix A is said to be singular if its determinant is zero. $|A| = 0$

A square matrix A is said to be non-singular if its determinant is not equal to zero. $|A| \neq 0$

11. Transpose Matrix:

The matrix obtained by interchanging rows and column of the matrix A is called the transpose of Matrix A. It is denoted by A^T (read as A transpose).

If A = $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

Then A^T = $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

12. Symmetric Matrix:

A square matrix is called a symmetric matrix. If $a_{ij} = a_{ji}$ for all i and j.

For example: $A = \begin{bmatrix} 1 & -5 & 3 \\ -5 & -2 & 4 \\ 3 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} x & a & b \\ a & y & c \\ b & c & z \end{bmatrix}$

13. Skew–symmetric Matrix:

A square matrix is said to be skew symmetric if for all I and j and a

For example: $X = \begin{bmatrix} 0 & -3 & 2 \\ 3 & 0 & 4 \\ -3 & 4 & 2 \end{bmatrix}$

14. Equal Matrix:

Two matrices are said to be equal if and only if they are comparable and each element of, one is equal to the corresponding element of the other.

For example: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \end{bmatrix}$

15. Orthogonal Matrix:

A square matrix A is said to be orthogonal if $AA^T = AT^A = I$

16. Sub – Matrix:

Let A be a given matrix. By deleting a few rows and columns we get a new matrix called the sub matrix of A and A is also a sub matrix of itself.

For example: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

The sub – matrices are $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, [1 2 3], [4 5 6], $\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$, $\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$, $[1\ \ 2], [1\ \ 3], [2\ \ 3], [4\ \ 6], [5\ \ 6], [1], [2], [3], [4], [5], [6]$.

17. Scalar Multiplication:

Multiplication of every element in a matrix by a constant is called scalar multiplication.

For example: A = $\begin{bmatrix} 2 & 5 \\ 6 & 8 \end{bmatrix}$ Then 2A = 2$\begin{bmatrix} 2 & 5 \\ 6 & 8 \end{bmatrix}$ = $\begin{bmatrix} 4 & 10 \\ 12 & 16 \end{bmatrix}$

Properties of a Matrix

i) If A and B are two matrices of order 'm' and 'n' then A + B = B + A.

ii) (A +B) + C = A + (B + C)

iii) K (A + B) = KA + KB.

iv) Existence of additive identity, that is, A + 0 = 0 + A = A

v) Existence of additive inverse, that is, A + (-A) = (-A) + A = 0

vi) Cancellation law A + C = B + C $\Rightarrow$ A = B

vii) Multiplication is distributive over addition, that is A (B + C) = AB + AC.

viii) Matrix multiplication is associative that is, (AB) C = A (BC)

ix) If matrices A and O are of the order m x n, then AO = O = OA.

x) Multiplication of a matrix by unit matrix is that matrix itself AI = IA = A.

Product of Matrices

Two matrices A and B can be multiplied if and only if the number of columns in A is equal to the number of rows in B. The product matrix is denoted by AB. In these case we say matrices A and B are compatible for multiplication and the product matrix AB has the same number of rows as A and the same number of columns as B. Thus if A is a m x n matrix and B is a n x p matrix then AB is a m x p matrix.

Multiplication of Matrices

Two matrices A and B can be multiplied if and only if the number of column in A is equal to the number of rows in B. The resultant matrix AB will have the same number of rows as A and the same number of columns as B.

A = $[\quad]_{m \times n}$ B = $[\quad]_{n \times p}$

$\therefore AB = [\quad]_{m \times p}$

Illustration:

If $A = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ 2 & 0 \\ 5 & 3 \end{bmatrix}$ find AB

Solution:

Here, columns of A = 3 and rows B is also 3. Hence matrix multiplication is possible.

$$AB = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix}_{2 \times 3} \begin{bmatrix} 4 & 1 \\ 2 & 0 \\ 5 & 3 \end{bmatrix}_{2 \times 2}$$

$$= \begin{bmatrix} (3 \text{ x } 4) + (4 \text{ x } 2) + (2 \text{ x } 5) & (3\text{x } 1) + (4 \text{ x } 0) + (2 \text{ x } 3) \\ (1 \text{ x } 4) + (2 \text{ x } 2) + (5 \text{ x } 5) & (1 \text{ x } 1) + (2 \text{ x } 0) + (5 \text{ x } 3) \end{bmatrix}$$

$$= \begin{bmatrix} (12 + 8 + 10) & (3 + 0 + 6) \\ (4 + 4 + 25) & (1 + 0 + 15) \end{bmatrix} = \begin{bmatrix} 30 & 9 \\ 33 & 16 \end{bmatrix}_{2 \times 2}$$

Illustration :

If A = [5 2 8] and $B = \begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix}$ find AB.

Solution:

Here, columns of A = 3 and rows of B = 3. Hence matrix multiplication is possible

$$AB = [5\ 2\ 8]_{1 \times 3} \begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix}_{3 \times 1}$$

= [(5x4) + (2x2) + (8x7)]

= [(20 + 4 + 56)

= [80]

Illustration:

If $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix}$ then verify whether AB = BA.

Solution:

$$AB = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} (2 \text{ x } 5) + (3 \text{ x } (-1)) & (2 \text{ x } (-2)) + (3 \text{ x } 6) \\ ((-1)\text{x } 5) + (4 \text{ x } (-1)) & ((-1) \text{ x } (-2)) + (4 \text{ x } 6) \end{bmatrix}$$

$$= \begin{bmatrix} 10-3 & -4+18 \\ -5-4 & 2+24 \end{bmatrix} = \begin{bmatrix} 7 & 14 \\ -9 & 26 \end{bmatrix} \qquad \text{.... (1)}$$

$$BA = \begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} (5 \text{ x } 2) + ((-2) \text{ x } (-1)) & (5 \text{ x } 3) + ((-2) \text{ x } 4) \\ ((-1)\text{x } 2) + (6 \text{ x } (-1)) & ((-1) \text{ x } 3) + (6 \text{ x } 4) \end{bmatrix}$$

$$= \begin{bmatrix} 10+2 & 15-8 \\ -2-6 & -3+24 \end{bmatrix} = \begin{bmatrix} 12 & 7 \\ -8 & 21 \end{bmatrix} \qquad \text{.... (2)}$$

From (1) and (2) we get AB $\neq$ BA

Thus it verified.

Illustration :

If A = $\begin{bmatrix} 17 & 5 \\ 14 & 6 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix}$ then find the value of x if $5x + 2B = A$.

Solution:

$$5x + 2B = A$$

$$5x = A - 2B$$

$$2B = 2\begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & -4 \end{bmatrix}$$

$$5x = A - 2B = \begin{bmatrix} 17 & 5 \\ 14 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} 17-2 & 5-0 \\ 14-4 & 6-(-4) \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ 10 & 10 \end{bmatrix}$$

$$\therefore \; x = \frac{1}{5}\begin{bmatrix} 15 & 5 \\ 10 & 10 \end{bmatrix} = \begin{bmatrix} \frac{15}{5} & \frac{5}{5} \\ \frac{10}{5} & \frac{10}{5} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$

$\therefore$ The value of $x = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$

Illustration :

Show that $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ satisfies the equation $A^2 - 4A - 5I = 0$.

Solution:

$$A^2 = AA = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$4A = 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$$

$$5I = 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^2 - 4A = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= 0$$

$\therefore A^2 - 4A - 5I = 0$. Thus it showed that A satisfied the equation.

Inverse of a Matrix

Finding the Inverse by Adjoint Matrix Method

$$A^{-1} = \frac{\mathbf{1}}{|\mathbf{A}|}\ \mathbf{adj\ A}$$

An Important Fact.

$\mathbf{A^{-1}}$ exists if and only if $|A| \neq 0$.

Illustration :

Find the inverse of $\begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$

Solution:

$$A^{-1} = \frac{\mathbf{1}}{|\mathbf{A}|}\ \mathbf{adj\ A}$$

$|A| = \begin{vmatrix} 2 & 2 \\ 3 & 5 \end{vmatrix} = 2 \times 5 - 2 \times 3 = 10 - 6 = 4 \neq 0$

$\therefore A^{-1}$ exists.

Adj A = (Cofactor of A)'

Cofactor of A:

$C_{11} = (-1)^{1+1}\, 5 = 5$

$C_{12} = (-1)^{1+2}\, 3 = -3$

$C_{21} = (-1)^{2+1}\, 2 = -2$

$C_{22} = (-1)^{2+2}\, 2 = 2$

$\therefore$ Cofactor of A $\begin{bmatrix} 5 & -3 \\ -2 & 2 \end{bmatrix}$

Adj A = (Cofactor of A)' = $\begin{bmatrix} 5 & -2 \\ -3 & 2 \end{bmatrix}$

$$A^{-1} = \frac{\mathbf{1}}{|\mathbf{A}|}\ \mathbf{adj\ A}$$

$$\mathbf{A^{-1}} = \frac{\mathbf{1}}{\mathbf{4}}\begin{bmatrix} 5 & -2 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & \frac{-2}{4} \\ \frac{-3}{4} & \frac{2}{4} \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & \frac{-1}{2} \\ \frac{-3}{4} & \frac{1}{2} \end{bmatrix}$$

Illustration :

Find the inverse of $\begin{bmatrix} 4 & 0 & 2 \\ 2 & 10 & 2 \\ 3 & 9 & 1 \end{bmatrix}$

Solution:

$$A^{-1} = \frac{\mathbf{1}}{|\mathbf{A}|}\ \mathbf{adj\ A}$$

$$|A| = \begin{vmatrix} 4 & 0 & 2 \\ 2 & 10 & 2 \\ 3 & 9 & 1 \end{vmatrix}$$

$$= 4\begin{vmatrix} 10 & 2 \\ 9 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 10 \\ 3 & 9 \end{vmatrix}$$

$= 4[(10 \text{ x } 1) – (9 \text{ x } 2)] – 0 + 2[(2 \text{ x } 9) – (3 \text{ x } 10)]$

$= 4(10 – 18) + 2(18 – 30)$

$= 4(- 8) + 2(-12)$

$= –32 – 24$

$= -56 \neq 0$

$\therefore A^{-1}$ exists.

Adj A = (Cofactor of A)'

Cofactor of A:

$C_{11} = (-1)^{1+1} \begin{vmatrix} 10 & 2 \\ 9 & 1 \end{vmatrix} = 10 - 18 = -8$

$C_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} = -(2 - 6) = +4$

$C_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 10 \\ 3 & 9 \end{vmatrix} = 18 - 30 = -12$

$C_{21} = (-1)^{2+1} \begin{vmatrix} 0 & 2 \\ 9 & 1 \end{vmatrix} = -(0 - 18) = +18$

$C_{22} = (-1)^{2+2} \begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} = 4 - 6 = -2$

$C_{23} = (-1)^{2+3} \begin{vmatrix} 4 & 0 \\ 3 & 9 \end{vmatrix} = -(36 - 0) = -36$

$C_{31} = (-1)^{3+1} \begin{vmatrix} 0 & 2 \\ 10 & 2 \end{vmatrix} = 0 - 20 = -20$

$C_{32} = (-1)^{3+2} \begin{vmatrix} 4 & 2 \\ 2 & 2 \end{vmatrix} = -(8 - 4) = -4$

$C_{33} = (-1)^{3+3} \begin{vmatrix} 4 & 0 \\ 2 & 10 \end{vmatrix} = 40 - 0 = +40$

Cofactor of A = $\begin{bmatrix} -8 & 4 & -12 \\ 18 & -2 & -36 \\ -20 & -4 & 40 \end{bmatrix}$

Adj A = (Cofactor of A)' = $\begin{bmatrix} -8 & 18 & 20 \\ 4 & -2 & -4 \\ -12 & -36 & 40 \end{bmatrix}$

$\mathbf{A^{-1} = \frac{1}{-56}} \begin{bmatrix} -8 & 18 & 20 \\ 4 & -2 & -4 \\ -12 & -36 & 40 \end{bmatrix}$

$= \begin{bmatrix} \frac{-8}{-56} & \frac{18}{-56} & \frac{20}{-56} \\ \frac{4}{-56} & \frac{-2}{-56} & \frac{-4}{-56} \\ \frac{-12}{-56} & \frac{-36}{-56} & \frac{40}{-56} \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{7} & \frac{-9}{28} & \frac{5}{14} \\ \frac{-1}{14} & \frac{1}{28} & \frac{1}{14} \\ \frac{3}{14} & \frac{9}{14} & \frac{-5}{7} \end{bmatrix}$$

Solution of a System of Linear Equations

By Matrix Inversion Technique

$$AX = B$$

$$X = A^{-1} B$$

Illustration :

Solve:

$$2x_1 + 3x_2 - x_3 = 9;$$

$$x_1 + x_2 + x_3 = 9;$$

$$3x_1 - x_2 - x_3 = -1$$

Solution:

$$AX = B$$

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix} \; X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \; B = \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$$

$$X = A^{-1} B$$

$$A^{-1} = \frac{\mathbf{1}}{|\mathbf{A}|} \textbf{ adj A}$$

$$|A| = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix}$$

$= 2[(1 \times -1) - (-1 \times 1)] - 3[(1 \times -1) - (3 \times 1)] \; -1[(1 \times -1) - (3 \times 1)]$

$= 2(-1 + 1) - 3(-1 - 3) -1(-1 - 3) = 2(0) -3(-4) -1(-4)$

$= 0 + 12 + 4 = 16 \neq 0$

$\therefore$ A^{-1} exists.

Adj A = (Cofactor of A)'

Cofactor of A:

$C_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} = 0$

$C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = +4$

$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = -4$

$C_{21} = (-1)^{2+1} \begin{vmatrix} 3 & -1 \\ -1 & -1 \end{vmatrix} = +4$

$C_{22} = (-1)^{2+2} \begin{vmatrix} 2 & -1 \\ 3 & -1 \end{vmatrix} = +1$

$C_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = +11$

$C_{31} = (-1)^{3+1} \begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} = +4$

$C_{32} = (-1)^{3+2} \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -3$

$C_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1$

Cofactor of A = $\begin{bmatrix} 0 & 4 & -4 \\ 4 & 1 & 11 \\ 4 & -3 & -1 \end{bmatrix}$

$\therefore$Adj A = (Cofactor of A)' = $\begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$

$\mathbf{A^{-1}} = \frac{\mathbf{1}}{\mathbf{16}} \begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$

$\mathbf{A^{-1}\,B} = \frac{\mathbf{1}}{\mathbf{16}} \begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix} \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$

$$= \frac{\mathbf{1}}{\mathbf{16}} \begin{bmatrix} 32 \\ 48 \\ 64 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{32}{16} \\ \frac{48}{16} \\ \frac{64}{16} \end{bmatrix}$$

But X = A^{-1} B. i.e., $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

$x_1 = 2$,

$x_2 = 3$ and

$x_3 = 4$.

Substitute these values of $x_1, x_2,$ *and* x_3 in the equations and see whether they are satisfied to ensure that the solution is correct.

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Practical Problems:

1. Find A+ B and A – B if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}; B = \begin{bmatrix} -1 & -2 & -4 \\ -1 & -2 & -4 \\ 1 & 2 & 4 \end{bmatrix}$$

2. Find AB and BA when

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} B = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$$

3. If $A = \begin{bmatrix} 2 & 5 & -1 \\ 3 & -1 & 2 \\ 7 & 2 & -3 \end{bmatrix}$ find A^{-1}

4. Solve by using matrix inversion method.

$$2x - y + 3z = 1;$$

$$x + y + z = 2;$$

$$x - y + z = 4.$$

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